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FACULTY OF SCIENCES
DEPARTMENT OF MATHEMATICS AND INFORMATICS

Smiljana Jakšić

Distributions and ultradistributions on $\mathbb{R}^d_+$ through Laguerre expansions with applications to pseudo-differential operators with radial symbols

-doctoral dissertation-

Смиљана Јакшић

Дистрибуције и ултрадистрибуције на $\mathbb{R}^d_+$ кроз Лагеров развој са применама на псеудо-диференцијалне операторе са радијалним симболима

-докторска дисертација-

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Dedicated to my loving parents and my dear sister

Contents

Preface

This dissertation is submitted for the degree of Doctor of Mathematics at the University of Novi Sad. The research described herein was conducted under the supervision of Professor Stevan Pilipović in the Department of Mathematics and Informatics, University of Novi Sad and Professor Bojan Prangoski in the Faculty of Mechanical Engineering, University Ss. Cyril and Methodius, between July 2014 and April 2016.

This work is to the best of my knowledge original except where references are made to previous work. Neither this nor any substantially similar dissertation has been submitted for any other diploma at any other university.

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Chapter 0

Introduction

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$, beside playing an important role in different fields of mathematics, was defined to extend the Fourier transform to the space of tempered distributions. However, physical considerations related to unrenormalizable and nonlocalizable field theories made it desirable to extend the Fourier transform to a larger class of functionals than tempered distributions. The Gelfand-Shilov spaces $S_\alpha(\mathbb{R}^d)$, $S^\beta(\mathbb{R}^d)$ and $S_\alpha^\beta(\mathbb{R}^d)$, $\alpha + \beta \geq 1$ arose (see [4], [12], [13], [15]-[18], [26]). The spaces $S_\alpha^\alpha(\mathbb{R}^d)$, $\alpha \geq 1/2$ are of the special interest, because as the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, they are invariant under the Fourier transform. The Hermite functions, which are an orthonormal basis for $L^2(\mathbb{R}^d)$ and eigenfunctions of the Fourier transform, have a special role for the characterization of not only the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and its dual space but also the Gelfand-Shipov spaces and their dual spaces i.e. all these spaces have been characterized in terms of the coefficients of their Fourier-Hermite expansions (see [3], [27], [31]).

In Chapter 2 we will consider the space of rapidly decreasing functions on $(0,\infty)^d$ i.e. $\mathcal{S}(\mathbb{R}_+^d)$ and its dual space $\mathcal{S}(\mathbb{R}_+^d)'$, that is, the space of tempered distributions supported by $[0,\infty)^d$. The problem of expanding the elements of $\mathcal{S}'(\mathbb{R}_+)$ with respect to the Laguerre orthonormal basis has been treated by M. Guillemont-Teissier in [19] and A. Duran in [5] (see also [32], [43] and [44]). The novelty of this thesis is the extension of the results of [19] for the d-dimensional case. As a consequence of this result, we obtain the Schwartz kernel theorem (Theorem 2.4.1) which states that there is one-to-one correspondence between elements from $\mathcal{S}'(\mathbb{R}_+^{m+n})$ in two sets of variables x and y and the continuous linear mappings of $(\mathcal{S}(\mathbb{R}_+^n))_y$ into $(\mathcal{S}'(\mathbb{R}_+^m))_x$. Also, as a an outcome we get that $\mathcal{S}'(\mathbb{R}_+^d)$ is a convolution algebra (see Remark 2.4.1).

The results concerning the extension of a smooth function out of some region and various reformulation of such problems are called extension theorems of Whitney type. One can see Whitney [41], Seeley [36] and Hörmander [20, Theorem 2.3.6, p. 48]. In Chapter 2.4 we solve a problem of an extension of a function from $\mathcal{S}(\mathbb{R}_+^d)$ onto $\mathcal{S}(\mathbb{R}^d)$ (Theorem 2.4.2).

Let us considered the analogous transform to the Fourier transform for the positive real line $(0,\infty)$, that is, the Hankel-Clifford transform. In Section 3.2 we define this transform for the d-dimensional case. Next, we show that it is a continuous mapping from $\mathcal{S}(\mathbb{R}_+^d)$ into the same space.

Thus, the following question arises: will there be analogous spaces associated with the Hankel-Clifford transform in the same way as the Gelfand-Shilov spaces are with the Fourier transform. In order to give an answer to this question, for $d = 1$, A. Duran introduced the G-type spaces i.e. $G_\alpha(\mathbb{R}^d_+)$, $G^\beta(\mathbb{R}^d_+)$ and $G^\beta_\alpha(\mathbb{R}^d_+)$, $\alpha + \beta \geq 2$ in [8]. In Chapter 3 we extend the definition of the G-type spaces for the d-dimensional case. Moreover, we introduce, as an important novelty of this thesis, the modified fractional power of the partial Hankel-Clifford transform. We prove that this transformation is a topological isomorphism on $G^\alpha_\alpha(\mathbb{R}^d_+)$ (Theorem 3.2.2).

Next, the Laguerre functions, which are an orthonormal basis for $L^2(\mathbb{R}^d_+)$ and eigenfunctions of the Hankel-Cliford transform have a similar role, as the Hermite functions have for the Gelfand-Shilov spaces, in the characterization of the G-type spaces. In Section 3.3 the characterization of elements from $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ through the Fourier-Laguerre coefficients estimate is given. Although, the paper [9] contains significant results on the characterization of the spaces $G^\alpha_\alpha(\mathbb{R}_+)$, $\alpha \geq 1$, we noticed subtle gaps which are improved in the d-dimensional case. The main corrections are related to the analytic function $F(w)$, $w \in \mathbf{D}$ (Proposition 3.3.3).

In Section 3.4 we describe the topological properties of $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$. Since the explanations for the one dimensional case given in [9] is inadequate. This is essentially improved in the multi-dimensional case (Theorems 3.4.1 and Theorem 3.4.2) by De Wilde's closed-graph theorem for ultrabornological spaces. Moreover, in Section 3.5 as a main consequence of the analysed topological structure we prove the Schwartz's kernel theorem $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and their dual spaces.

In Section 3.6 we use the expansion of the Laguerre functions into finite sums of the Hermite functions and vice versa in order to prove that there exists a topological isomorphism between $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and the subspaces of the Gelfand-Shilov spaces $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, $\alpha \geq 1$ consisting of "even" functions denoted as $\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d)$.

Furthermore, in Section 3.7 we give two structural theorems for $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, $\alpha \geq 1$ (Theorem 3.7.1, Theorem 3.7.2). The first one states that $f \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$, $\alpha \geq 1$ if and only if it can be written as

$$f = \left(\sum_{k \in \mathbb{N}^d_0} c_k \left(xD^2 + D - \frac{x}{4} + \frac{1}{2} \right)^k \right) F$$

where $F \in L^2(\mathbb{R}^d_+)$ and the coefficients c_k have a suitable growth. $(xD^2 + D - x/4 + 1/2)^k = \prod_{j=1}^d (x_j D_j^2 + D_j - x_j/4 + 1/2)^{k_j}$, $k \in \mathbb{N}^d_0$. The second one is similar to the first one, but instead of using the operator $(xD^2 + D - x/4 + 1/2)^k$, $f \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$ is represented as an infinite sum of integrals of $L^2(\mathbb{R}^d_+)$-functions integrated against the test functions that are differentiated and then multiplied by powers of x suitable number of times.

In Section 4.4, we use the obtained series expansions for $G^\alpha_\alpha(\mathbb{R}^d)$, $\alpha \geq 1$ and their dual spaces in order to introduce a new class of pseudo-differential operators with radial symbols and prove continuity properties of such operators on the Gelfand- Shilov spaces and their dual spaces. More precisely, we prove the continuity of the Weyl pseudo-differential operators with radial symbols from the spaces $G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)$ and $(G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$, $\alpha \geq 1/2$. In the first case, we show that the class of the Weyl pseudo-differential operators with radial symbols is a continuous and linear mapping from $S^{\alpha}_{\alpha}(\mathbb{R}^d)$ into $S^{\alpha}_{\alpha}(\mathbb{R}^d)$ which can be extended to a continuous and linear mapping from $(S^{\alpha}_{\alpha}(\mathbb{R}^d))'$ into $S^{\alpha}_{\alpha}(\mathbb{R}^d)$. In the second case, we show that the class of the Weyl pseudo-differential operators with radial symbols is a continuous and linear mapping from $S^{\alpha}_{\alpha}(\mathbb{R}^d)$ into $S^{\alpha}_{\alpha}(\mathbb{R}^d)$ which can be extended to a continuous and linear mapping from $(S^{\alpha}_{\alpha}(\mathbb{R}^d))'$ into $(S^{\alpha}_{\alpha}(\mathbb{R}^d))'$. This second case is especially important since the symbols are in the dual spaces and the corresponding mapping is over the dual spaces of the Gelfand-Shilov spaces. As a remark (Remark 4.4.1), we have shown that this symbol class is in the bijection with the space $(\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d))'$, closely related to dual spaces of "even" Gelfand-Shilov spaces.

Finally, In Section 4.5, we give the corresponding results related to radial symbols in $\mathcal{S}(\mathbb{R}^d_+)$ and its dual space and the corresponding continuous linear mappings related to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and its dual space. With these special cases, we extend the corresponding results of M. W. Wong [42, Chapter 24].

Chapter 1

Notation and background

1.1 Euclidean Spaces

Let $\mathbb{R}^d$ be the usual Euclidean space given by

$$\mathbb{R}^d = \{(x_1, ..., x_d) : x_j\text{'s are real numbers}\}.$$

Let $x = (x_1, ..., x_d)$ and $y = (y_1, ..., y_d)$ be any two points in $\mathbb{R}^d$. The inner product $x \cdot y$ of x and y is defined by

$$x \cdot y = \sum_{j=1}^{d} x_j y_j$$

and the norm $|x|$ of x is defined by

$$|x| = \Big(\sum_{j=1}^{d} x_j^2\Big)^{\frac{1}{2}}.$$

The symbol $\mathbb{R}^d_+$ stands for $(0, \infty)^d$ i.e.

$$\mathbb{R}^d_+ = \{(x_1, ..., x_d) : x_j\text{'s are real numbers greater than zero}\}$$

and $\overline{\mathbb{R}^d_+}$ for its closure i.e. $[0, \infty)^d$. We denote the set of all real numbers by $\mathbb{R}$, the set of all positive integers by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, the set of all non-negative integers ≥ 0 by $\mathbb{N}_0$ and the set of all complex numbers by $\mathbb{C}$.

1.2 The Multi-index Notation

We use the standard multi-index notation. We denote by $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$. Thus, for $z \in \mathbb{C}^d$, $z^{\mathbf{1}}$ stands for $z_1 \cdot \ldots \cdot z_d$. A d-dimensional multi-index is a d-tuple $\alpha = (\alpha_1, ..., \alpha_d)$ of non-negative integers. We call $|\alpha| = \sum_{j=1}^{d} \alpha_j$ the length of the multi-index α. For multi-index $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ (or $x \in \overline{\mathbb{R}^d_+}$), we denote the power by

$$x^\alpha = x_1^{\alpha_1} ... x_d^{\alpha_d}$$

and the partial derivative by

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

Furthermore, if $x, \gamma \in \overline{\mathbb{R}_+^d}$ we also use

$$x^\gamma = x_1^{\gamma_1} \ldots x_d^{\gamma_d}.$$

In this case, if $x_j = 0$ and $\gamma_j = 0$, we use the convention $0^0 = 1$.

1.3 The Sequence spaces

A complex sequence $\{a_n\}_{n \in \mathbb{N}_0^d}$ is said to be rapidly decreasing if for every constant $j \geq 0$ the quantity

$$\sup_{n \in \mathbb{N}_0^d} |a_n|(|n| + 1)^j \tag{1.1}$$

is finit. The rapidly decreasing sequences form a vector space, which we denote by s and on which we put the topology defined by the seminorms (1.1) for $j = 0, 1, 2, \ldots$. It is easy to check that s is a Fréchet space. From now on, we abbreviate a Fréchet space as an (F)-space.

Theorem 1.3.1. *([39, Theorem 51.5]) The Fréchet space s of rapidly decreasing sequences is nuclear.*

From now on, we abbreviate nuclear Fréchet space as an (FN)-space. Clearly, in the definition of s, we can take the l^p-norms, $p \geq 1$ instead of the sup-norm. In this case, a complex sequence $\{a_n\}_{n \in \mathbb{N}_0^d}$ is said to be rapidly decreasing if for every constant $j \geq 0$ the quantity

$$\Big(\sum_{n \in \mathbb{N}_0^d} (|a_n|(|n| + 1)^j)^p \Big)^{1/p}$$

is finite.

A sequence $\{b_n\}_{n \in \mathbb{N}_0^d}$ is said to be slowly growing if there is a constant $j \geq 0$ such that

$$\sup_{n \in \mathbb{N}_0^d} |b_n|(|n| + 1)^{-j} < \infty.$$

It can easily be verified that the mapping

$$\{b_n\}_{n \in \mathbb{N}_0^d} \to (\{a_n\}_{n \in \mathbb{N}_0^d} \to \sum_{n \in \mathbb{N}_0^d} a_n b_n)$$

is an isomorphism (for the vector space structures) of the space of slowly growing sequences (which we shall denote by s') onto the dual of s. The space s' is equipped with the strong dual topology of s; s' is a strong dual of a nuclear Fréchet space (abbreviated as a (DFN)-space).

Let $\alpha \geq 1$ and $a > 1$. We define $s^{\alpha,a}$ to be the space of all complex sequences $\{a_n\}_{n\in\mathbb{N}_0^d}$ for which

$$\|\{a_n\}_{n\in\mathbb{N}_0^d}\|_{s^{\alpha,a}} = \sup_{n\in\mathbb{N}_0^d} |a_n| a^{|n|^{1/\alpha}} < \infty.$$

With this norm $s^{\alpha,a}$ becomes a Banach space (abbreviated as a (B)-space). For $a > b > 1$, $s^{\alpha,a}$ is continuously injected into $s^{\alpha,b}$. As a locally convex space (abbreviated as an l.c.s.) we define

$$s^\alpha = \varinjlim_{a\to 1^+} s^{\alpha,a}.$$

Note that s^α, as an inductive limit of $s^{\alpha,a}$, is indeed a (Hausdorff) l.c.s. since $s^{\alpha,a}$ is continuously injected into s.

Proposition 1.3.1. *For $a > b > 1$, the canonical inclusion $s^{\alpha,a} \to s^{\alpha,b}$ is nuclear. In particular, s^α is a nuclear (DFS)-space (i.e. a (DFN)-space) and its strong dual $(s^\alpha)'$ is an (FN)-space.*

Proof. Since the canonical inclusion $s^{\alpha,a} \to s^{\alpha,b}$ is a composition of two inclusions of the same type it is enough to prove that it is quasinuclear (for the definition of a quasinuclear mapping see Definition A.1.2 and for the fact that the composition of two quasinuclear mappings is nuclear see Theorem A.1.1. For each $m \in \mathbb{N}_0^d$, we define $e_m \in (s^{\alpha,a})'$ by

$$\langle e_m, \{a_n\}_{n\in\mathbb{N}_0^d}\rangle = a_m b^{|m|^{1/\alpha}}.$$

One easily verifies that

$$\|e_m\|_{(s^{\alpha,a})'} \leq (b/a)^{|m|^{1/\alpha}}.$$

Hence,

$$\sum_{m\in\mathbb{N}_0^d} \|e_m\|_{(s^{\alpha,a})'} < \infty.$$

For $\{a_n\}_{n\in\mathbb{N}_0^d} \in s^{\alpha,a}$ we have

$$\|\{a_n\}_{n\in\mathbb{N}_0^d}\|_{s^{\alpha,b}} \leq \sum_{m\in\mathbb{N}_0^d} |a_m| b^{|m|^{1/\alpha}} = \sum_{m\in\mathbb{N}_0^d} |\langle e_m, \{a_n\}_{n\in\mathbb{N}_0^d}\rangle|,$$

i.e. the canonical inclusion $s^{\alpha,a} \to s^{\alpha,b}$ is quasinuclear. □

For the moment, denote by $\tilde{s}^\alpha$ the space of all complex valued sequences $\{b_n\}_{n\in\mathbb{N}_0^d}$ such that for each $a > 1$,

$$\|\{b_n\}_{n\in\mathbb{N}_0^d}\|_{\tilde{s}^\alpha,a} = \sum_{n\in\mathbb{N}_0^d} |b_n| a^{-|n|^{1/\alpha}} < \infty.$$

With these seminorms $\tilde{s}^\alpha$ becomes an (F)-space. Denote by Ξ the mapping

$$\tilde{s}^\alpha \to (s^\alpha)'\, , \langle \Xi(\{b_n\}_n), \{a_n\}_n\rangle = \sum_n a_n b_n.$$

One easily verifies that it is a well defined bijection. Let $B \subseteq \tilde{s}^{\alpha}$ be bounded. If $B_1 \subseteq s^{\alpha}$ is bounded, there exists $a > 1$ such that $B_1 \subseteq s^{\alpha,a}$ and it is bounded there (s^{α} is a (DFN)-space). Now one easily verifies that

$$\sup_{\{b_n\}_n \in B,\, \{a_n\}_n \in B_1} |\langle \Xi(\{b_n\}_n), \{a_n\}_n \rangle| < \infty,$$

i.e. Ξ maps bounded sets into bounded. Since $\tilde{s}^{\alpha}$ and $(s^{\alpha})'$ are (F)-spaces, Ξ is continuous and now the open mapping theorem verifies that Ξ is an isomorphism. Hence, we proved the following result.

Proposition 1.3.2. *The strong dual $(s^{\alpha})'$ of s^{α} is an (FN)-space of all complex valued sequences $\{b_n\}_{n \in \mathbb{N}_0^d}$ such that, for each $a > 1$,*

$$\|\{b_n\}_{n \in \mathbb{N}_0^d}\|_{(s^{\alpha})',a} = \sum_{n \in \mathbb{N}_0^d} |b_n| a^{-|n|^{1/\alpha}} < \infty.$$

Its topology is generated by the system of seminorms $\|\cdot\|_{(s^{\alpha})',a}$.

1.4 Laguerre functions and $L^2(\mathbb{R}^d_+, x^{\gamma}dx)$

Let $\gamma \in \overline{\mathbb{R}^d_+}$. We denote by $L^2(\mathbb{R}^d_+, x^{\gamma}dx)$ the space of all measurable functions on $\mathbb{R}^d_+$ for which

$$\int_{\mathbb{R}^d_+} |f(x)|^2 x^{\gamma} dx < \infty.$$

Its norm is defined by the square root of the last quantity.

Let $\gamma \geq 0$. The one-dimensional Laguerre polynomials of order γ are defined by

$$L_n^{\gamma}(x) = \frac{x^{-\gamma}e^x}{n!}\Big(\frac{d}{dx}\Big)^n (e^{-x}x^{\gamma+n}), \qquad x \geq 0,\ n \in \mathbb{N}_0.$$

The corresponding Laguerre functions of order γ are given by

$$\mathcal{L}_n^{\gamma}(x) = \Big(\frac{n!}{\Gamma(n+\gamma+1)}\Big)^{\frac{1}{2}} L_n^{\gamma}(x) e^{-\frac{x}{2}}, \qquad x \geq 0,\ n \in \mathbb{N}_0.$$

In the case $\gamma = 0$, we write L_n and $\mathcal{L}_n$ instead of L_n^0 and $\mathcal{L}_n^0$, respectively.

Now, we list properties we will need in the sequel:

(i) Let $\gamma \in \overline{\mathbb{R}^d_+}$. The d-dimensional Laguerre functions of order γ are products of the one-dimensional Laguerre functions of order γ; namely,

$$\mathcal{L}_n^{\gamma}(x) = \mathcal{L}_{n_1}^{\gamma_1}(x_1) \dots \mathcal{L}_{n_d}^{\gamma_d}(x_d), \qquad x \in \overline{\mathbb{R}^d_+},\ n \in \mathbb{N}_0^d.$$

(ii) $\{\mathcal{L}_n^{\gamma}\}_{n \in \mathbb{N}_0^d}$ is an orthonormal basis for $L^2(\mathbb{R}^d_+, x^{\gamma}dx)$.

(iii) The operator
$$E_\gamma = \frac{d}{dx}\Big(x\frac{d}{dx}\Big) - \frac{x}{4} - \frac{\gamma^2}{4x} + \frac{\gamma+1}{2} \tag{1.2}$$
is called the Laguerre operator. Notice that E is a self-adjoint operator, i.e.
$$\langle Ef, g\rangle = \langle f, Eg\rangle \text{ for } f,g \in \mathrm{dom}(E) = \{f \in L^2(\mathbb{R}^d_+);\ Ef \in L^2(\mathbb{R}^d_+)\}.$$
Now we state an important property of this operator (see [11, (11), p. 188]):
$$E_\gamma(x^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x)) = -nx^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x), \qquad x > 0. \tag{1.3}$$
Hence, the Laguerre functions $\mathcal{L}_n(x)$ are the eigenfunctions of the operator E.

Next, we define the d-dimensional Laguerre operator
$$E_\gamma = \prod_{i=1}^d \Big(\frac{d}{\partial x_j}\Big(x_j\frac{\partial}{\partial x_j}\Big) - \frac{x_j}{4} - \frac{\gamma_j^2}{4x_j} + \frac{\gamma_j+1}{2}\Big). \tag{1.4}$$
Now, from (i) and (1.3) follows that $x^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x)$ is an eigenfunction of the Laguerre operator in each variable. Hence,
$$E_\gamma(x^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x)) = -|n|x^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x), \qquad x \in \mathbb{R}^d_+.$$
In the case $\gamma = 0$, we write E instead of E_γ. Notice that the d-dimensional Laguerre functions $\mathcal{L}_n(x)$ are the eigenfunctions of the operator E.

Remark 1.4.1. The d-dimensional Laguerre operator can be also defined by
$$E_\gamma = \prod_{i=1}^d \Big(\frac{d}{\partial x_j}\Big(x_j\frac{\partial}{\partial x_j}\Big) - \frac{x_j}{4} - \frac{\gamma_j^2}{4x_j}\Big).$$
Then
$$E_\gamma(x^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x)) = -\Big|n + \frac{\gamma+1}{2}\Big|x^{\frac{\gamma}{2}}\mathcal{L}_n^\gamma(x), \qquad x \in \mathbb{R}^d_+.$$

(iv) We have the following inequality for the Laguerre polynomials (see [11, (3), p.205])
$$e^{-\frac{x}{2}}|L_n(x)| \le 1, \qquad x \ge 0. \tag{1.5}$$

(v) We have the following estimate for the derivatives of the Laguerre polynomials of order γ.

Theorem 1.4.1. *([6, Theorem 1])*
$$\Big|x^k\frac{d^p}{dx^p}(e^{-x/2}L_n^\gamma(x))\Big|$$
$$\le 2^{-\min\{\gamma,k\}}4^k(n+1)\cdot\ldots\cdot(n+k)\binom{n+\max\{\gamma-k,0\}+p}{n},$$
for all $x \ge 0$, $n,k,p \in \mathbb{N}_0$.

Note, for $\gamma = 0$ in the previous theorem, we obtain

$$\left| x^{(p+k)/2} \frac{d^k}{dx^k} (e^{-x/2} L_n^{\gamma}(x)) \right| \\ \leq 2^{p+k+5} (n+1) \cdot \ldots \cdot (n + \left[\frac{p+k}{2}\right] + 2) \binom{n+k}{n}. \tag{1.6}$$

Bounds in Theorem 1.4.1 can be improved for certain values of k. Indeed, taking $k = p/2$, we obtain

Lemma 1.4.1. *([9, Lemma 2.1.]) If $p, n \in \mathbb{N}$, $x > 0$ and $\gamma > 0$ then*

$$\left| x^{p/2} \frac{d^p}{dx^p} (e^{-x/2} L_n^{\gamma}(x)) \right| \\ \leq 32(p+4)(n+1) \cdot \ldots \cdot (n + \left[\frac{p}{2}\right] + 3)(n+1) \binom{n+\gamma}{n}.$$

In the proof of the previous lemma the following estimate was shown

$$\left| x^{(p+k)/2} \frac{d^p}{dx^p} (e^{-x/2} L_n^{\gamma}(x)) \right| \\ \leq 2 \cdot 4^{k+2} (n+1) \cdot \ldots \cdot (n + \left[\frac{p+k}{2}\right] + 2) \binom{n+k}{n}. \tag{1.7}$$

In [19], p. 547 the following bound on the Laguerre functions is proved:

$$\left| x^k \left(\frac{d}{dx}\right)^p \mathcal{L}_n(x) \right| \leq C_{p,k} (n+1)^{p+k}, \tag{1.8}$$

for all $x \geq 0$, $n, k, p \in \mathbb{N}_0$.

(vi) The Laguerre polynomials satisfy a simple integral equation with a symmetric kernel (see [25, (4.20.3), p. 83]):

$$\int_0^{\infty} J_{\gamma}(\sqrt{xt}) x^{\gamma/2} L_n^{\gamma}(x) dx = 2(-1)^n x^{\gamma/2} L_n^{\gamma}(x), \; \gamma \geq 0, \; n \in \mathbb{N}_0, \tag{1.9}$$

where J_{γ} is the Bessel function of the first kind.

(vii) We have the following recurrence formula (see [11, (24), p.190])

$$L_n^{\gamma-1}(x) = L_n^{\gamma}(x) - L_{n-1}^{\gamma}(x). \tag{1.10}$$

(viii) We can represent the Laguerre polynomials as finite sums (see [11, (41), p. 192])

$$L_n^{\alpha+\beta+1}(x+y) = \sum_{k=0}^{n} L_k^{\alpha}(x) L_{n-k}^{\beta}(y). \tag{1.11}$$

and (see [11, (39), p. 192])

$$L_n^{\alpha}(t) = \sum_{n=0}^{\infty} (m!)^{-1} (\alpha - \beta)_m L_{n-m}^{\beta}(t). \tag{1.12}$$

(ix) The Laplace transform of $t^{\gamma} L_n^{\gamma}(t)$ is given by (see [11, p. 191])

$$\int_0^{\infty} t^{\gamma} L_n^{\gamma}(t) e^{-st} dt = \frac{\Gamma(n+1+\gamma)(s-1)^n}{n! s^{\gamma+n+1}}, \; \gamma > -1, \; \mathrm{Re}\, s > 0. \tag{1.13}$$

1.5 Hermite functions and $L^2(\mathbb{R}^d)$

We denote by $L^2(\mathbb{R}^d)$ the space of all measurable functions on $\mathbb{R}^d$ for which

$$\int_{\mathbb{R}^d} |f(x)|^2 dx < \infty.$$

Its norm is defined by the square root of the last quantity.

The one-dimensional Hermite polynomials are given by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dt^n}(e^{-x^2}), \qquad x \in \mathbb{R},\ n \in \mathbb{N}_0.$$

The Hermite functions are given by

$$h_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x), \qquad x \in \mathbb{R},\ n \in \mathbb{N}_0.$$

Now, we list properties we will need in the sequel:

(i) The d-dimensional Hermite functions are product of the one-dimensional Hermite functions; namely,

$$h_n(x) = h_{n_1}(x_1) \dots h_{n_d}(x_d).$$

(ii) $\{h_n\}_{n \in \mathbb{N}_0^d}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.

(iii) The operator $H = x^2 - (d^2/dx^2)$ is called the Hermite operator. The one-dimensional Hermite functions are the eigenfunctions of this operator:

$$\left(x^2 - \frac{d^2}{dx^2}\right) h_n = (2n+1) h_n.$$

In d dimensions, this equation together with (i) shows that h_n is an eigenfunction of the Hermite operator in each variable,

$$\left(x_j^2 - \frac{d^2}{dx_j^2}\right) h_n = (2n_j+1) h_n,$$

as well as of the d-dimensional Hermite operator

$$(x^2 - D^2) h_n = \prod_{j=1}^{d} \left(x_j^2 - \frac{\partial^2}{\partial x_j^2}\right) h_n = (2|n|+1) h_n.$$

(iv) The Hermite polynomials can be expressed in terms of the Laguerre polynomials (see [11, (2), p. 193])

$$H_{2n}(x) = (-1)^m 2^{2m} m! L_m^{-\frac{1}{2}}(x^2), \qquad x \in \mathbb{R},\ n \in \mathbb{N}_0 \tag{1.14}$$

and

$$H_{2n+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{\frac{1}{2}}(x^2), \qquad x \in \mathbb{R},\ n \in \mathbb{N}_0. \tag{1.15}$$

These expressions show that $H_n(x)$ is an even function or an odd function of x according as n is even or odd.

(v) H. Cramér has proved the following bound for the Hermite polynomials (see [11, (19), p. 208])

$$e^{-\frac{1}{2}x^2}|H_n(x)| < k2^{\frac{n}{2}}(n!)^{\frac{1}{2}}, \qquad x \in \mathbb{R},\ n \in \mathbb{N}_0, \tag{1.16}$$

where the constant k is less than 1.0864435.

1.6 The Function Spaces $S_\alpha^\alpha(\mathbb{R}^d)$

Problems of regularity of solutions to partial differential equations (PDEs) play a central role in the modern theory of PDEs. When a solution of a certain PDE is smooth but not analytic, we seek to find a space where we can describe its decay for $|x| \to \infty$ and regularity in $\mathbb{R}^d$. Gelfand and Shilov introduced the space of type S in order to find solutions of certain parabolic initial-value problems.

We denote the set of all infinitely differential functions on $\mathbb{R}^d$ by $\mathcal{C}^\infty(\mathbb{R}^d)$.

The Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is the set of all $\mathcal{C}^\infty$ function φ on $\mathbb{R}^d$ such that

$$\sup_{x\in\mathbb{R}^d} |x^m D^n \varphi(x)| < \infty, \qquad n, m \in \mathbb{N}_0^d.$$

Let $\alpha \geq 1/2$. For $A > 0$, denote by $\mathcal{S}_{\alpha,A}^{\alpha,A}(\mathbb{R}^d)$ a Banach space (abbreviated as a (B)-space) of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with the norm

$$\sup_{n,m\in\mathbb{N}_0^d} \frac{\|x^m D^n \varphi(x)\|_{L^2(\mathbb{R}^d)}}{(A^{|n|+|m|} n!^\alpha m!^\alpha)} < \infty.$$

The Gelfand-Shilov spaces $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ are inductive limits of the spaces $S_{\alpha,A}^{\alpha,A}(\mathbb{R}^d)$ with respect to A:

$$S_\alpha^\alpha(\mathbb{R}^d) = \varinjlim_{A\to\infty} S_{\alpha,A}^{\alpha,A}(\mathbb{R}^d).$$

The space $S_\alpha^\alpha(\mathbb{R}^d)$ is nontrivial if and only if $\alpha \geq 1/2$. When the spaces are nontrivial we have a dense and continuous inclusion:

$$S_\alpha^\alpha(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d).$$

The corresponding dual spaces of $S_\alpha^\alpha(\mathbb{R}^d)$ are the spaces of ultradistributions of Roumier type:

$$(S_\alpha^\alpha(\mathbb{R}^d))' = \varprojlim_{A\to 0} (S_{\alpha,A}^{\alpha,A}(\mathbb{R}^d))'.$$

One easily verifies that for $A_1 < A_2$, the canonical inclusion

$$\mathcal{S}_{\alpha,A_1}^{\alpha,A_1}(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{\alpha,A_2}^{\alpha,A_2}(\mathbb{R}^d)$$

is a compact mapping, i.e. $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ is a strong dual of a Fréchet-Schwartz space (abbreviated as a (DFS)-space). For the properties of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, we refer to [26, Chapter 6]; see also [12], [18].

For each $n \in \mathbb{N}_0^d$, $h_n \in \mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$. Moreover, for $\alpha \geq 1/2$, $S_\alpha^\alpha(\mathbb{R}^d)$ is given through the Hermite expansions. Let $\iota : \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \to s^{2\alpha}$, $\iota(f) = \{\langle f, h_n\rangle\}_{n\in\mathbb{N}_0^d}$. It is proved that ι is a well defined bijection between $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ and $s^{2\alpha}$ (see [3], [10]). Moreover, we have the following results for which the proof is similar to the proof of [24, Theorem 3.4 and Corollary 3.5] and we omit it.

Theorem 1.6.1. *Let $\alpha \geq 1/2$. The mapping*

$$\iota :\to s^{2\alpha},\ \iota(f) = \{\langle f, h_n\rangle\}_{n\in\mathbb{N}_0^d}$$

is a topological isomorphism between $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ and $s^{2\alpha}$.

For each $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, $\sum_{n\in\mathbb{N}_0^d}\langle f, h_n\rangle h_n$ converges absolutely to f in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$.

Theorem 1.6.2. *Let $\alpha \geq 1/2$. The mapping*

$$\tilde{\iota} : (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \to (s^{2\alpha})',\ \tilde{\iota}(T) = \{\langle T, h_n\rangle\}_{n\in\mathbb{N}_0^d}$$

is a topological isomorphism.

Moreover, for each $T \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$, $\sum_{n\in\mathbb{N}_0^d}\langle T, h_n\rangle h_n$ converges absolutely to T in $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$.

1.7 The Function Spaces $G_\alpha^\beta(\mathbb{R}_+)$

In this section, we state the results obtained by A. Duran in [8] and [9]. We were motivated by these papers. Our goal was to extend the results for the d dimensional case. Although, it may look trivial, it was booth technical and mathematical demanding. Also, we found subtle gaps in the proofs which we corrected.

The Hankel-Clifford transform $\mathcal{H}_0$ is defined by

$$\mathcal{H}_0(f)(t) = \frac{1}{2}\int_0^\infty f(x)J_0(\sqrt{xt})dx,$$

where J_0 denotes the Bessel function of the first kind. The Hankel-Clifford transform is analogous to the Fourier transform for the positive real line $(0,\infty)$ and is an isomorphism from the Schwartz space defined on $(0,\infty)$ (denoted by $\mathcal{S}(\mathbb{R}_+)$) onto itself (see [43]). In [8] A. Duran introduced the spaces of test functions $G_\alpha(\mathbb{R}_+)$, $G^\beta(\mathbb{R}_+)$ and $G_\alpha^\beta(\mathbb{R}_+)$ for $\alpha, \beta \geq 0$ in order to define the Hankel-Clifford transform on a larger class of functionals than tempered distributions with positive support.

Definition 1.7.1. ([8, Definition 2.1.]) Let $\alpha, \beta \geq 0$ and $A, B > 0$. The spaces $G_{\alpha,A}(\mathbb{R}_+)$, $G^{\beta,B}(\mathbb{R}_+)$ and $G_{\alpha,A}^{\beta,B}(\mathbb{R}_+)$ are defined as the set of all complex valued $\mathcal{C}^\infty$ functions f belongs $\mathcal{S}(\mathbb{R}_+)$ and satisfy:

$$\forall\delta > 0\ \forall p \in \mathbb{N}\ \exists C_{\delta,p} > 0\ \forall k \in \mathbb{N} : \|t^{(p+k)/2}f^{(p)}(t)\|_2 \leq C_{\delta,p}(A+\delta)^k k^{(\alpha/2)k}$$

$$\forall\varrho > 0\ \forall k \in \mathbb{N}\ \exists C_{\varrho,k} > 0\ \forall p \in \mathbb{N} : \|t^{(p+k)/2}f^{(p)}(t)\|_2 \leq C_{\varrho,k}(B+\varrho)^p p^{(\beta/2)p}$$

$$\begin{aligned}&\forall\delta, \varrho > 0\ \exists C_{\delta,\varrho} > 0\ \forall k, p \in \mathbb{N} :\\ &\|t^{(p+k)/2}f^{(p)}(t)\|_2 \leq C_{\delta,\varrho}(A+\delta)^k(B+\varrho)^p k^{(\alpha/2)k}p^{(\beta/2)p},\end{aligned}$$

respectively.

The spaces $G_\alpha(\mathbb{R}_+)$, $G^\beta(\mathbb{R}_+)$ and $G_\alpha^\beta(\mathbb{R}_+)$ are defined as the union of the spaces $G_{\alpha,A}(\mathbb{R}_+)$ when $A>0$, $G^{\beta,B}(\mathbb{R}_+)$ when $B>0$ and $G_{\alpha,A}^{\beta,B}(\mathbb{R}_+)$ when $A,B>0$, respectively.

Let f be from $\mathcal{S}(\mathbb{R}_+)$. The modified fractional power of the Hankel-Clifford transform is defined as

$$\mathcal{J}_{z,\gamma}(f)(t) = \frac{1}{1-z}\int_0^\infty (xtz)^{-\gamma/2}x^\gamma I_\gamma\Big(\frac{2\sqrt{xtz}}{1-z}\Big)f(x)dx,$$

where I_γ is the modified Bessel function of the first kind and $z \in \mathbb{C}$, $|z|=1$, $z \neq 1$. When $z=-1$ we obtain the Hankel-Clifford transform. The following lemma is the key to prove that the Hankel-Clifford transform is an isomorphism from the spaces $G_\alpha(\mathbb{R}_+)$, $G^\beta(\mathbb{R}_+)$ and $G_\alpha^\beta(\mathbb{R}_+)$ onto $G^\alpha(\mathbb{R}_+)$, $G_\beta(\mathbb{R}_+)(\mathbb{R}_+)$ and G_β^α, respectively.

Lemma 1.7.1. *([8, Lemma 3.2]) Let f be from $\mathcal{S}(\mathbb{R}_+)$, $\gamma>-1$, $z\in\mathbb{C}$, $|z|=1$, $z\neq 1$ and $p,k>0$. Then*

$$\left\|t^{(p+k+\gamma)/2}f^{(p)}(t)\right\|_2 = |1-z|^{-p+k}\left\|t^{(p+k+\gamma)/2}\mathcal{J}_{z,\gamma}^{(k)}f(t)\right\|_2. \tag{1.17}$$

The following result about non triviality of the spaces $G_\alpha(\mathbb{R}_+)$, $G^\beta(\mathbb{R}_+)$ and $G_\alpha^\beta(\mathbb{R}_+)$ is obtained.

Corollary 1.7.1. *([8, Corollary 3.9.]) For every $\alpha,\beta\geq 0$, the spaces $G_\alpha(\mathbb{R}_+)$, $G^\beta(\mathbb{R}_+)$ are nontrivial. The space $G_\alpha^\beta(\mathbb{R}_+)$ reduces to the null-function if and only if*

(i) $\beta=0$ *and* $\alpha\leq 2$

(ii) $\alpha=0$ *and* $\beta\leq 2$

(iii) $\alpha\neq 0$, $\beta\neq 0$ *and* $\alpha+\beta<2$.

The characterization of the spaces $G_\alpha^\alpha(\mathbb{R}_+)$ in terms of their Fourier-Laguerre coefficients is given in [9].

Theorem 1.7.1. *([9, Theorem 3.6.]) Let $f\in L^2(0,\infty)$, $\alpha\geq 1$ and*

$$a_n = \int_0^\infty f(t)L_n(t)e^{-t/2}dt.$$

The following conditions are equivalent

(i) There exist two constants $c>0$ and $a>1$ such that

$$|a_n|\leq ca^{-n^{1/\alpha}}, \quad \textit{for } n\geq 0.$$

(ii) The function $f\in G_\alpha^\alpha(\mathbb{R}_+)$.

Conversely, given a sequence $(a_n)_n$ satisfying the contdition (i) there exists $f \in G_\alpha^\alpha(\mathbb{R}_+)$ such that $a_n = \int_0^\infty f(t)L_n(t)e^{-t/2}dt$ for $n \in \mathbb{N}$.

Two integral transforms play a fundamental role in the proof of these results. One of them is the above mentioned Hankel-Clifford transform and the other is the Fourier-Laplace type operator $\mathcal{F}_D$ defined in the space $G_\alpha^\alpha(\mathbb{R}_+)$ by

$$\mathcal{F}_D(f)(w) = \int_0^\infty f(t)e^{-\frac{1}{2}\frac{1+w}{1-w}t}dt, \quad \text{for } w \in D$$

where D is the unite disc. In [7], it was shown that $\mathcal{F}_D(f)(w) = (1-w)\sum_n a_n w^n$ (see Proposition 3.2). In order to prove Theorem 1.7.1, firstly a characterization of the analytic functions on the unite disc was given.

Lemma 1.7.2. *([9, Lemma 3.2.]) Let $F \in H(D)$ and $\alpha \geq 0$. If we put $F(w) = (1-w)\sum_n a_n w^n$, then the following conditions are equivalent:*

(i) There exists constants $C, A > 0$ such that

$$|F^{(p)}(w)| \leq CA^p p^{\alpha p}, \quad \textit{for } p \geq 0 \textit{ and } w \in D.$$

(ii) There exist constants $c > 0$ i $a > 1$ such that

$$|a_n| \leq ca^{-n^{1/\alpha}}, \quad \textit{for } n \geq 0.$$

Secondly, the following corollary was proved.

Corollary 1.7.2. *([9, Corollary 3.5.]) Let $f \in G_\alpha^\alpha$ and $\alpha \geq 1$. Then there exists constants $C, A > 0$ such that*

$$|(\mathcal{F}_D(f))^{(p)}(w)| \leq CA^p p^{\alpha p} \quad \textit{for all } p \geq 0 \textit{ and } w \in D.$$

We finish this section with generalization of the previous theorem for the dual spaces $(G_\alpha^\alpha(\mathbb{R}_+))'$, $\alpha \geq 1$.

Corollary 1.7.3. *([9, Corollary 3.8.]) We define the operator $L_\gamma : (G_\alpha^\alpha(\mathbb{R}_+))' \to \mathbb{C}^\mathbb{N}$ by $L_\gamma(u) = (\langle u, \mathcal{L}_n^\gamma \rangle)_n$. Then the mapping*

$$L_\gamma : (G_\alpha^\alpha(\mathbb{R}_+))' \to \{(a_n)_n \ : \ \forall a > 1, \ \|(a_n)_n\|_a = \sup_n\{|a_n a^{-n^{1/\alpha}}|\} < \infty\}$$

is an isomorphism between these spaces.

Chapter 2

The Function Space $\mathcal{S}(\mathbb{R}^d_+)$

In this chapter we consider the space $\mathcal{S}(\mathbb{R}^d_+)$ which consists of all $f \in \mathcal{C}^\infty(\mathbb{R}^d_+)$ such that all derivatives $D^p f$, $p \in \mathbb{N}_0^d$, extend to continuous functions on $\overline{\mathbb{R}^d_+}$ and

$$\sup_{x \in \mathbb{R}^d_+} x^k |D^p f(x)| < \infty \ , \forall k, p \in \mathbb{N}_0^d.$$

With this system of seminorms $\mathcal{S}(\mathbb{R}^d_+)$ becomes an (F)-space. We denote by $\mathcal{S}'(\mathbb{R}^d_+)$ its strong dual.

Firstly, we will show that the topology on $\mathcal{S}(\mathbb{R}^d_+)$ can be given by the L^2-norms.

Secondly, we will show that the mapping $\iota : \mathcal{S}(\mathbb{R}^d_+) \to s,\ \iota(f) = \{a_n(f)\}_{n \in \mathbb{N}_0^d}$, where $a_n(f) = \int_{\mathbb{R}^d_+} f(x)\mathcal{L}_n(x)dx$, is a topological isomorphisam between $\mathcal{S}(\mathbb{R}^d_+)$ and s. In [5] and [19] was proved, for one dimensional case, that ι is a well defined bijection.

Thirdly, we give a characterization of $\mathcal{S}'(\mathbb{R}^d_+)$ in terms of the Fourier-Laguerre coefficients. As a consequence we obtain that $\mathcal{S}'(\mathbb{R}^d_+)$ is topologically isomorphic to s'. Also we show that $\mathcal{S}'(\mathbb{R}^d_+)$ is a convolution algebra.

Finally, the Schwartz kernel theorems for both $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ will be given. As a consequence, we will obtain the extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}^d_+)$.

2.1 Another definition of the space $\mathcal{S}(\mathbb{R}^d_+)$

Using the Sobolev embedding theorem (see Theorem C.0.1), we will prove that the topology of $\mathcal{S}(\mathbb{R}^d_+)$ can be also defined by the L^2-seminorms instead of the supremum seminorms. We need to verify that $\mathbb{R}^d_+$ satisfies the strong local Lipschitz condition (see Definition C.0.1) in order to obtain the assertion. For the moment, denote $\mathbf{C} = \mathbb{R}^d_+$. On the hyperplane $x_1 + \ldots + x_d = 0$ take $d-1$ orthonormal vectors $\xi_1, \ldots, \xi_{d-1}$ and let $\xi_d = (-1/\sqrt{d}, \ldots, -1/\sqrt{d})$ (given in the $x_1, \ldots, x_d$ coordinate system). Then, $\xi_1, \ldots, \xi_d$ is an orthonormal basis for $\mathbb{R}^d$. Notice that the boundary of $\mathbf{C}$ is exactly the graph, given in the $(\xi_1, \ldots, \xi_d)$-coordinate system of a continuous piecewise linear function f in $\xi_1, \ldots, \xi_{d-1}$ such that the domain of each piece is a polyhedral cone. Thus, this function is Lipschitz continuous on $\mathbb{R}^{d-1}$ and $\mathbf{C}$ is represented by the inequality $\xi_d < f(\xi_1, \ldots, \xi_{d-1})$. This proves that $\mathbf{C} = \mathbb{R}^d_+$ satisfies the strong local Lipschitz condition. Thus, the Sobolev embedding theorem is applicable on $\mathbb{R}^d_+$, i.e. for all $j \in \mathbb{N}_0$, the Sobolev space $W^{j+j_0}(\mathbb{R}^d_+)$ is continuously injected into $\mathcal{C}^j(\overline{\mathbb{R}^d_+})$, where $2j_0 > d \geq 2(j_0-1)$ (here, $\mathcal{C}^j(\overline{\mathbb{R}^d_+})$ denotes a (B)-space of all functions which have bounded uniformly continuous derivatives up to order j; the norm is given by $\sup_{|k|\leq j}\sup_{x\in\mathbb{R}^d_+}|D^k\varphi(x)|$). This implies that the topology on $\mathcal{S}(\mathbb{R}^d_+)$ can be given by the family of seminorms

$$\varphi \mapsto \left(\sum_{|k|\leq j,\,|p|\leq j} \|x^k D^p \varphi\|^2_{L^2(\mathbb{R}^d_+)}\right)^{1/2}, \quad j \in \mathbb{N}_0. \tag{2.1}$$

2.2 Convergence of the Laguerre series in $\mathcal{S}(\mathbb{R}^d_+)$

Theorem 2.2.1. *For $f \in \mathcal{S}(\mathbb{R}^d_+)$ let $a_n(f) = \int_{\mathbb{R}^d_+} f(x)\mathcal{L}_n(x)dx$. Then*

$$f = \sum_{n\in\mathbb{N}_0^d} a_n(f)\mathcal{L}_n$$

and the series converges absolutely in $\mathcal{S}(\mathbb{R}^d_+)$.

Moreover, the mapping

$$\iota : \mathcal{S}(\mathbb{R}^d_+) \to s,\ \iota(f) = \{a_n(f)\}_{n\in\mathbb{N}_0^d},$$

is a topological isomorphism.

Proof. Let E be the Laguerre operator. By Remark 1.4.1, for $f \in \mathcal{S}(\mathbb{R}^d_+)$

$$a_n(Ef) = \langle Ef, \mathcal{L}_n\rangle = \langle f, E(\mathcal{L}_n)\rangle = a_n(f)\prod_{i=1}^d -\Big(n_i + \frac{1}{2}\Big).$$

Moreover,

$$a_n(E^p f) = a_n(f)\prod_{i=1}^d (-1)^{p_i}(n_i + \frac{1}{2})^{p_i}, \text{ for any } p \in \mathbb{N}^d.$$

As $E^p f \in \mathcal{S}(\mathbb{R}^d_+) \subset L^2(\mathbb{R}^d_+)$, we have

$$\sum_{n\in\mathbb{N}_0^d} |a_n(f)|^2 \prod_{i=1}^d \Big(n_i + \frac{1}{2}\Big)^{2p_i} < \infty, \text{ for all } p \in \mathbb{N}_0^d,$$

i.e. $\{a_n(f)\}_{n\in\mathbb{N}_0^d} \in s$. Clearly $f = \sum_{n\in\mathbb{N}_0^d} a_n(f)\mathcal{L}_n$ as elements of $L^2(\mathbb{R}^d_+)$. By (1.8), we find the bound on the d-dimensional Laguerre functions without complicated calculation:

$$|x^k D^p \mathcal{L}_n(x)| \leq C_{p,k}\prod_{i=1}^d (n_i+1)^{p_i+k_i},\ x \in \mathbb{R}^d_+,\ n,p,k \in \mathbb{N}_0^d.$$

Hence, we obtain

$$\sum_{n\in\mathbb{N}_0^d} |x^k D^p(a_n(f)\mathcal{L}_n(x))| \le C_{p,k} \sum_{n\in\mathbb{N}_0^d} |a_n(f)| \prod_{i=1}^{d} (n_i+1)^{p_i+k_i} < \infty \tag{2.2}$$

which yields the absolute convergence of the series in $\mathcal{S}(\mathbb{R}_+^d)$.

To prove that ι is a topological isomorphism, firstly observe that by the above computations it is well defined and it is clearly an injection. Let $\{a_n\}_{n\in\mathbb{N}_0^d} \in s$. Define $f = \sum_{n\in\mathbb{N}_0^d} a_n\mathcal{L}_n \in L^2(\mathbb{R}_+^d)$. Now (2.2) proves that this series converges in $\mathcal{S}(\mathbb{R}_+^d)$, hence $f \in \mathcal{S}(\mathbb{R}_+^d)$. Thus ι is bijective. Observe that, (2.2) proves that ι^{-1} is continuous. Since $\mathcal{S}(\mathbb{R}_+^d)$ and s are (F)-spaces, the open mapping theorem proves that ι is a topological isomorphism (see Apendix A.2). $\square$

2.3 Convergence of the Laguerre series in $\mathcal{S}'(\mathbb{R}_+^d)$

Theorem 2.3.1. *For $T \in \mathcal{S}'(\mathbb{R}_+^d)$ let $b_n(T) = \langle T, \mathcal{L}_n\rangle$. Then*

$$T = \sum_{n\in\mathbb{N}_0^d} b_n(T)\mathcal{L}_n$$

and $\{b_n(T)\}_{n\in\mathbb{N}_0^d} \in s'$. The series converges absolutely in $\mathcal{S}'(\mathbb{R}_+^d)$.

Conversely, if $\{b_n\}_{n\in\mathbb{N}_0^d} \in s'$ then there exists $T \in \mathcal{S}'(\mathbb{R}_+^d)$ such that $T = \sum_{n\in\mathbb{N}_0^d} b_n\mathcal{L}_n$.

As a consequence, $\mathcal{S}'(\mathbb{R}_+^d)$ is topologically isomorphic to s'.

Proof. Let $\{b_n\}_{n\in\mathbb{N}_0^d} \in s'$. There exists $k \in \mathbb{N}$ such that $\sum_{n\in\mathbb{N}_0^d} |b_n|^2(|n|+1)^{-2k} < \infty$. For a bounded subset B of $\mathcal{S}(\mathbb{R}_+^d)$, Theorem 2.2.1 implies that there exists $C > 0$ such that

$$\sum_{n\in\mathbb{N}_0^d} |a_n(f)|^2(|n|+1)^{2k} \le C,\ \forall f \in B,$$

where we denote $\{a_n(f)\}_{n\in\mathbb{N}_0^d} = \iota(f)$. Observe that for arbitrary $q \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{|n|\le q} \sup_{f\in B} |\langle b_n\mathcal{L}_n, f\rangle| &\le \sup_{f\in B} \sum_{n\in\mathbb{N}_0^d} \sum_{m\in\mathbb{N}_0^d} |\langle b_n\mathcal{L}_n, a_m(f)\mathcal{L}_m\rangle| \\ &= \sup_{f\in B} \sum_{n\in\mathbb{N}_0^d} |b_n||a_n(f)| \le C',\end{aligned}$$

i.e.

$$\sum_{n\in\mathbb{N}_0^d} \sup_{f\in B} |\langle b_n\mathcal{L}_n, f\rangle| < \infty.$$

Hence, $\sum_{n\in\mathbb{N}_0^d} b_n\mathcal{L}_n$ converges absolutely in $\mathcal{S}'(\mathbb{R}_+^d)$.

Let $T \in \mathcal{S}'(\mathbb{R}_+^d)$. Theorem 2.2.1 implies that ${}^t\iota : s' \to \mathcal{S}'(\mathbb{R}_+^d)$ is an isomorphism (${}^t\iota$ denotes the transpose of ι). Now, one easily verifies that

$$({}^t\iota)^{-1}T = \{b_n\}_{n\in\mathbb{N}_0^d},$$

where $b_n(T) = \langle T, \mathcal{L}_n \rangle$. Observe that for $f \in \mathcal{S}(\mathbb{R}^d_+)$

$$\langle T, f \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) \langle T, \mathcal{L}_n \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) b_n(T) = \left\langle \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n, f \right\rangle,$$

i.e. $T = \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n$. □

Remark 2.3.1. ([5, Remark 3.7 for d=1]) Let us show that $\mathcal{S}'(\mathbb{R}^d_+)$ is a convolution algebra. Given $f, g \in \mathcal{S}'(\mathbb{R}^d_+)$, we compute the n-th Laguerre coefficient of $f * g$ if $a_n = \langle f, \mathcal{L}_n \rangle$ and $b_n = \langle g, \mathcal{L}_n \rangle$ then

$$\langle f * g, \mathcal{L}_n(t) \rangle = \langle f(x) \otimes g(y), \mathcal{L}_n(x + y) \rangle.$$

In order to simplify the proof, we consider the case $d = 2$. Using (1.10) and (1.11), we obtain

$$\begin{aligned}
\langle f * g, \mathcal{L}_n(t) \rangle &= \langle f(x) \otimes g(y), \prod_{i=1}^{2} (\mathcal{L}^1_{n_i}(x_i + y_i) - \mathcal{L}^1_{n_i - 1}(x_i + y_i)) \rangle \\
&= \langle f(x) \otimes g(y), \prod_{i=1}^{2} \Big(\sum_{k_i=0}^{n_i} \mathcal{L}_{n_i - k_i}(x_i) \mathcal{L}_{k_i}(y_i) - \sum_{k_i=0}^{n_i - 1} \mathcal{L}_{n_i - k_i - 1}(x_i) \mathcal{L}_{k_i}(y_i) \Big) \rangle \\
&= \langle f(x) g(y), \sum_{k \le (n_1, n_2)} \mathcal{L}_{(n_1, n_2) - k}(x) \mathcal{L}_k(y) - \sum_{k \le (n_1 - 1, n_2)} \mathcal{L}_{(n_1 - 1, n_2) - k}(x) \mathcal{L}_k(y) \\
&\quad - \sum_{k \le (n_1, n_2 - 1)} \mathcal{L}_{(n_1, n_2 - 1) - k}(x) \mathcal{L}_k(y) + \sum_{k \le (n_1 - 1, n_2 - 1)} \mathcal{L}_{(n_1 - 1, n_2 - 1) - k}(x) \mathcal{L}_k(y) \rangle \\
&= \sum_{k \le (n_1, n_2)} a_{(n_1, n_2) - k} b_k - \sum_{k \le (n_1 - 1, n_2)} a_{(n_1 - 1, n_2) - k} b_k \\
&\quad - \sum_{k \le (n_1, n_2 - 1)} a_{(n_1, n_2 - 1) - k} b_k + \sum_{k \le (n_1 - 1, n_2 - 1)} a_{(n_1 - 1, n_2 - 1) - k} b_k,
\end{aligned}$$

where a_n or b_n equals zero if some component of the subindex n is less than zero. It is easy to verify that if $(a_n)_{n \in \mathbb{N}^2} \in s'$ and $(b_n)_{n \in \mathbb{N}^2} \in s'$ then $\langle f * g, \mathcal{L}_n(t) \rangle \in s'$.

2.4 Kernel theorem for $\mathcal{S}(\mathbb{R}^m_+)$ and its dual space. The Extension Theorem of Whitney type

In this section we will prove that spaces $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ are nuclear. This fact, together with Theorem 2.2.1 will lead us to the kernel theorem of Schwartz.

For the review of the toplogical tensor product we refer to Appendix A.5.

Proposition 2.4.1. *The spaces $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ are nuclear.*

Proof. Since s is nuclear Theorem 2.2.1 implies that $\mathcal{S}(\mathbb{R}^d_+)$ is also nuclear. From Proposition A.5.3 follows that $\mathcal{S}'(\mathbb{R}^d_+)$ is nuclear as the strong dual of a nuclear (F)-space. □

Theorem 2.4.1. *The following canonical isomorphisms hold:*

$$\mathcal{S}(\mathbb{R}^m_+)\hat{\otimes}\mathcal{S}(\mathbb{R}^n_+) \cong \mathcal{S}(\mathbb{R}^{m+n}_+)$$

and

$$S'(\mathbb{R}^m_+)\hat{\otimes}\mathcal{S}'(\mathbb{R}^n_+) \cong \mathcal{S}'(\mathbb{R}^{m+n}_+) \cong \mathcal{L}(\mathcal{S}(\mathbb{R}^n_+), \mathcal{S}'(\mathbb{R}^m_+)). \tag{2.3}$$

Proof. The second isomorphism follows from the first since $\mathcal{S}(\mathbb{R}^d_+)$ is a nuclear (F)-space. Thus it is enough to prove the first isomorphism.

Step 1: From Theorem 2.2.1 follows that $\mathcal{S}(\mathbb{R}^m_+) \otimes \mathcal{S}(\mathbb{R}^n_+)$ is dense in $\mathcal{S}(\mathbb{R}^{m+n}_+)$. It suffices to show that the latter induces on the former the topology $\pi = \epsilon$ (the π and the ϵ topologies are the same because $\mathcal{S}(\mathbb{R}^d_+)$ is nuclear). Since the bilinear mapping $(f, g) \mapsto f \otimes g$ of $\mathcal{S}(\mathbb{R}^m_+) \times \mathcal{S}(\mathbb{R}^n_+)$ into $\mathcal{S}(\mathbb{R}^{m+n}_+)$ is separately continuous it follows that it is continuous ($\mathcal{S}(\mathbb{R}^m_+)$ and $\mathcal{S}(\mathbb{R}^n_+)$ are (F)-spaces). The continuity of this bilinear mapping proves that the inclusion $\mathcal{S}(\mathbb{R}^m_+) \otimes_\pi \mathcal{S}(\mathbb{R}^n_+) \to \mathcal{S}(\mathbb{R}^{m+n}_+)$ is continuous, hence the topology π is stronger than the induced one from $\mathcal{S}(\mathbb{R}^{m+n}_+)$ onto $\mathcal{S}(\mathbb{R}^m_+) \otimes \mathcal{S}(\mathbb{R}^n_+)$.

Step 2: Let A' and B' be equicontinuous subsets of $\mathcal{S}'(\mathbb{R}^m_+)$ and $\mathcal{S}'(\mathbb{R}^n_+)$, respectively. There exist $C > 0$ and $j, l \in \mathbb{N}$ such that such that

$$\sup_{T\in A'} |\langle T, \varphi\rangle| \le C\|\varphi\|_{j,l} \quad \text{and} \quad \sup_{F\in B'} |\langle F, \psi\rangle| \le C\|\psi\|_{j,l},$$

where

$$\|f\|_{j,l} = \sup_{\substack{|k|\le j \\ |p|\le l}} \sup_{x\in\mathbb{R}^d_+} |x^k D^p f(x)| < \infty. \tag{2.4}$$

For all $T \in A'$ and $F \in B'$ we have

$$\begin{aligned}
|\langle T_x \otimes F_y, \chi(x, y)\rangle| &= |\langle F_y, \langle T_x, \chi(x, y)\rangle\rangle| \le C \sup_{\substack{|k|\le j \\ |p|\le l}} \sup_{y\in\mathbb{R}^n_+} |y^k \langle T_x, D^p_y \chi(x, y)\rangle| \\
&\le C^2 \sup_{\substack{|k|\le j \\ |p|\le l}} \sup_{\substack{|k'|\le j \\ |p'|\le l}} \sup_{\substack{x\in\mathbb{R}^m_+ \\ y\in\mathbb{R}^n_+}} |x^{k'} y^k D^{p'}_x D^p_y \chi(x, y)| \\
&\le C^2 \|\chi(x, y)\|_{(k',k),(p',p)}, \ \forall \chi \in \mathcal{S}(\mathbb{R}^m_+) \otimes \mathcal{S}(\mathbb{R}^n_+).
\end{aligned}$$

It follows that the ϵ topology on $\mathcal{S}(\mathbb{R}^m_+) \otimes \mathcal{S}(\mathbb{R}^n_+)$ is weaker than the induced one from $\mathcal{S}(\mathbb{R}^{m+n}_+)$. $\square$

The isomorphism (2.3) calls for some comment. To every kernel $K(x, y) \in \mathcal{S}'(\mathbb{R}^{m+n}_+)$ we may associate a continuous linear mapping K of $\mathcal{S}(\mathbb{R}^n_+)$ into $\mathcal{S}'(\mathbb{R}^m_+)$ in the following manner: if $v \in \mathcal{S}(\mathbb{R}^n_+)$ then

$$(Kv)(x) = \int_{\mathbb{R}^n_+} K(x, y)v(y)dy \in \mathcal{S}'(\mathbb{R}^m_+).$$

Theorem 2.4.1 states that the correspondence $K(x, y) \leftrightarrow K$ is an isomorphism.

As a consequence of previous theorem we obtain the following important theorem.

Theorem 2.4.2. *The restriction mapping $f \mapsto f_{|\mathbb{R}^d_+}$, $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d_+)$ is a topological homomorphism onto.*

The space $\mathcal{S}(\mathbb{R}^d_+)$ is topologically isomorphic to the quotient space $\mathcal{S}(\mathbb{R}^d)/N$, where $N = \{f \in \mathcal{S}(\mathbb{R}^d) | \operatorname{supp} f \subseteq \mathbb{R}^d \backslash \mathbb{R}^d_+\}$. Consequently, $\mathcal{S}'(\mathbb{R}^d_+)$ can be identified with the closed subspace of $\mathcal{S}'(\mathbb{R}^d)$ which consists of all tempered distributions with support in $\overline{\mathbb{R}^d_+}$.

In order to prove this result, we need the theorem on the tensor product of linear mappings :

Theorem 2.4.3. *([23, Theorem 7, p.189]) Let A_1, A_2 be homomorphisms of E_1, E_2 onto dense subspace of F_1 and F_2, respectively. Then $A_1 \otimes_\pi A_2$ and $A_1 \hat{\otimes}_\pi A_2$ are homomorphisms of $E_1 \otimes_\pi E_2$, $E_1 \hat{\otimes}_\pi E_2$ onto dense subspace of $F_1 \otimes_\pi F_2$, $F_1 \hat{\otimes}_\pi F_2$, respectively.*

If E_1, E_2 are metrizable and A_1 and A_2 homomorphisms onto F_1 and F_2, respectively, then $A_1 \hat{\otimes}_\pi A_2$ is a homomorphism onto $F_1 \hat{\otimes}_\pi F_2$.

and the theorem about the duality of Fréchet-Schwartz spaces (abbreviated as an (FS)-space):

Theorem 2.4.4. *([28, Theorem A.6.5, p.255]) Let E be an (FS)-space and F be a closed subspace of E. Then E/F is a (FS)-space. Moreover, we have the following isomorphism of linear topological spaces*

$$(E/F)' \cong F^\perp,$$

where $F^\perp = \{x' \in E'; \langle x', y\rangle = 0 \text{ for any } y \in F\}$.

Now, we proceed to the proof of Theorem 2.4.2.

Proof. Obviously, the restriction mapping $f \mapsto f_{|\mathbb{R}^d_+}$, $\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d_+)$ is continuous. We prove its surjectivity by induction on d. For clarity, denote the d-dimensional restriction by R_d. For $d = 1$, the surjectivity of R_1 is proved in [5, p. 168]. Assume that R_d is surjective. By the open mapping theorem, R_d and R_1 are topological homomorphisms onto since all the underlying spaces are (F)-spaces. By the above theorem $R_d \hat{\otimes}_\pi R_1$ is continuous mapping from $\mathcal{S}(\mathbb{R}^{d+1})$ to $\mathcal{S}(\mathbb{R}^{d+1}_+)$ ($\mathcal{S}(\mathbb{R}^d)\hat{\otimes}\mathcal{S}(\mathbb{R}) \cong \mathcal{S}(\mathbb{R}^{d+1})$ by the Schwartz kernel theorem). Clearly $R_d \hat{\otimes}_\pi R_1 = R_{d+1}$. As $\mathcal{S}(\mathbb{R}^{d+1})$ and $\mathcal{S}(\mathbb{R}^{d+1}_+)$ are (F)-spaces Theorem 2.4.3 implies that R_{d+1} is also surjective.

The surjectivity of the restriction mapping together with the open mapping theorem implies that it is homomorphism. Clearly N is closed subspace of $\mathcal{S}(\mathbb{R}^d)$ and $\ker R_d = N$. Thus R_d induces natural topological isomorphism between $\mathcal{S}(\mathbb{R}^d)/N$ and $\mathcal{S}(\mathbb{R}^d_+)$. Hence $\big(\mathcal{S}(\mathbb{R}^d)/N\big)'_b$ is topologically isomorphic to $\mathcal{S}'(\mathbb{R}^d_+)$ (the index b stands for the strong dual topology). Since $\mathcal{S}(\mathbb{R}^d)$ is an (FS)-space, Theorem 2.4.4 implies that $\big(\mathcal{S}(\mathbb{R}^d)/N\big)'_b$ is topologically isomorphic to the closed subspace

$$N^\perp = \{T \in \mathcal{S}'(\mathbb{R}^d) | \, \langle T, f\rangle = 0, \ \forall f \in N\}$$

of $\mathcal{S}'(\mathbb{R}^d)$ which is exactly the subspace of all tempered distributions with support in $\overline{\mathbb{R}^d_+}$. □

Remark 2.4.1. The fact that $(\mathcal{S}(\mathbb{R}^d_+))'$ is canonically isomorphic to the closed subspace of $(\mathcal{S}(\mathbb{R}^d))'$ which consists of all tempered distributions with support in $\overline{\mathbb{R}^d_+}$ allows us to define unambiguously the notion of derivatives of the elements of $(\mathcal{S}(\mathbb{R}^d_+))'$. In fact, for $T \in (\mathcal{S}(\mathbb{R}^d_+))'$ and $n \in \mathbb{N}^d_0$, D^nT stands for the D^n-derivative of T in $(\mathcal{S}(\mathbb{R}^d))'$ sense. Since $\operatorname{supp} D^nT \subseteq \overline{\mathbb{R}^d_+}$, D^nT is a well defined element of $(\mathcal{S}(\mathbb{R}^d_+))'$. Moreover, by $\mathcal{S}(\mathbb{R}^d_+) \cong \mathcal{S}(\mathbb{R}^d)/N$ (see Theorem 2.4.2)

$$\langle D^nT, \varphi\rangle = (-1)^{|n|}\langle T, D^n\varphi\rangle, \;\; \forall \varphi \in \mathcal{S}(\mathbb{R}^d_+).$$

It is important to stress that if T is given by $\psi \in \mathcal{S}(\mathbb{R}^d_+)$ then D^nT does not have to coincide with the classical D^n-derivative of ψ (unless ψ can be extended to a smooth function on $\mathbb{R}^d$ with support in $\overline{\mathbb{R}^d_+}$). Considering ψ as an element of $(\mathcal{S}(\mathbb{R}^d_+))'$ automatically means extending it by 0 on $\mathbb{R}^d \backslash \overline{\mathbb{R}^d_+}$. Of course, this extension does not have to be smooth.

Chapter 3

The Function Spaces $G_\alpha^\beta(\mathbb{R}_+^d)$, $\alpha, \beta \geq 0$

In this chapter we consider the test spaces $G_\alpha^\beta(\mathbb{R}_+^d)$, $\alpha, \beta \geq 0$ i.e. the G-type spaces, for the spaces of ultradistributions supported by $[0, \infty)^d$.

Firstly, we will extend the Definition 1.7.1 for the d-dimensional case.

Secondly, we will define the fractional powers and the modified fractional powers of the Hankel-Clifford transform, denoted by $\mathcal{I}_{z,\gamma}$ and $\mathcal{J}_{z,\gamma}$, respectively, on $\mathcal{S}(\mathbb{R}_+^d)$. As a novelty of this thesis, we will introduce the modified fractional power of the partial Hankel-Clifford transform $\mathcal{J}_{z',\gamma'}^{(d')}$ on $\mathcal{S}(\mathbb{R}_+^d)$. We will prove that $\mathcal{I}_{z,\gamma}$, $\mathcal{J}_{z,\gamma}$ and $\mathcal{J}_{z',\gamma'}^{(d')}$ are topological isomorphisms on $\mathcal{S}(\mathbb{R}_+^d)$ and that they extend to isometries from $L^2(\mathbb{R}_+^d, t^\gamma dt)$ onto itself. We will also see the action of these transforms on the G-type spaces.

Thirdly, we will show that for $\alpha \geq 1$, $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \to s^\alpha$, $\iota(f) = \{\langle f, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$ (the definition of s^α is given in Section 1.3) is a well defined bijection. In this way we will extend the results of A. Duran in [9] for the d-dimensional case and we will do corrections of subtle gaps he made. Furthermore, we will prove that ι is a topological isomorphisam between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and s^α. Also, we will provide the similar results for the dual spaces $(G_\alpha^\alpha(\mathbb{R}_+^d))'$.

Fourthly, we will study the relation between the G-type spaces and the Gelfand-Shilov spaces. We will establish an existence of a topological isomorphism between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$ consisting of all "even" functions from $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$, where $\alpha \geq 1$.

Finally, we give two structural theorems for $(G_\alpha^\alpha(\mathbb{R}_+^d))'$, $\alpha \geq 1$.

3.1 Definition of $G_\alpha^\beta(\mathbb{R}_+^d)$, $\alpha, \beta \geq 0$

We define the basic test spaces following Definition 1.7.1 for $d = 1$ and we consider their topological structures.

Unless otherwise stated, α and β are two reals such that $\alpha, \beta \geq 0$.

Let $A>0$. We denote by $G^{\beta,A}_{\alpha,A}(\mathbb{R}^d_+)$ the space of all $f\in\mathcal{S}(\mathbb{R}^d_+)$ for which

$$\sup_{p,k\in\mathbb{N}^d_0}\frac{\|t^{(p+k)/2}D^pf(t)\|_2}{A^{|p+k|}k^{(\alpha/2)k}p^{(\beta/2)p}}<\infty.$$

With the following seminorms

$$\sigma_{A,j}(f)=\sup_{p,k\in\mathbb{N}^d_0}\frac{\|t^{(p+k)/2}D^pf(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{|p+k|}k^{(\alpha/2)k}p^{(\beta/2)p}}+\sup_{\substack{|p|\le j\\|k|\le j}}\sup_{t\in\mathbb{R}^d_+}|t^kD^pf(t)|,\ j\in\mathbb{N}_0,$$

one easily verifies that it becomes an (F)-space. Clearly, if $A_1<A_2$, $G^{\beta,A_1}_{\alpha,A_1}(\mathbb{R}^d_+)$ is continuously injected into $G^{\beta,A_2}_{\alpha,A_2}(\mathbb{R}^d_+)$. We define $G^\beta_\alpha(\mathbb{R}^d_+)$ as an inductive limit of the spaces $G^{\beta,A}_{\alpha,A}(\mathbb{R}^d_+)$ with respect to A:

$$G^\beta_\alpha(\mathbb{R}^d_+)=\varinjlim_{A\to\infty}G^{\beta,A}_{\alpha,A}(\mathbb{R}^d_+).$$

Since all the injections $G^{\beta,A}_{\alpha,A}\to\mathcal{S}(\mathbb{R}^d_+)$ are continuous, $G^\beta_\alpha(\mathbb{R}^d_+)$ is indeed a (Hausdorff) l.c.s.. Clearly, $G^\beta_\alpha(\mathbb{R}^d_+)$ is continuously injected into $\mathcal{S}(\mathbb{R}^d_+)$. As inductive limit of an (F)-spaces, $G^\beta_\alpha(\mathbb{R}^d_+)$ is a barrelled and bornological l.c.s..

For $A>0$ we define $G_{\alpha,A}(\mathbb{R}^d_+)$ to be the space of all $f\in\mathcal{S}(\mathbb{R}^d_+)$ such that

$$\sup_{k\in\mathbb{N}^d_0}\frac{\|t^{(p+k)/2}D^pf(t)\|_2}{A^{|k|}k^{(\alpha/2)k}}<\infty,\ \forall p\in\mathbb{N}^d_0$$

and similarly, $G^{\beta,A}(\mathbb{R}^d_+)$ to be the space of all $f\in\mathcal{S}(\mathbb{R}^d_+)$ such that

$$\sup_{p\in\mathbb{N}^d_0}\frac{\|t^{(p+k)/2}D^pf(t)\|_2}{A^{|p|}p^{(\beta/2)p}}<\infty,\ \forall k\in\mathbb{N}^d_0.$$

If we equip $G_{\alpha,A}(\mathbb{R}^d_+)$ with the system of seminorms

$$\sigma'_{A,j}(f)=\sup_{|p|\le j}\sup_{k\in\mathbb{N}^d_0}\frac{\|t^{(p+k)/2}D^pf(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{|k|}k^{(\alpha/2)k}}+\sup_{\substack{|p|\le j\\|k|\le j}}\sup_{t\in\mathbb{R}^d_+}|t^kD^pf(t)|,\ j\in\mathbb{N}_0,$$

one easily verifies that it becomes an (F)-space. Analogously, by equipping $G^{\beta,A}(\mathbb{R}^d_+)$ with the system of seminorms

$$\sigma''_{A,j}(f)=\sup_{|k|\le j}\sup_{p\in\mathbb{N}^d_0}\frac{\|t^{(p+k)/2}D^pf(t)\|_{L^2(\mathbb{R}^d_+)}}{A^{|p|}p^{(\beta/2)p}}+\sup_{\substack{|p|\le j\\|k|\le j}}\sup_{t\in\mathbb{R}^d_+}|t^kD^pf(t)|,\ j\in\mathbb{N}_0,$$

it is also an (F)-space. We define $G_\alpha(\mathbb{R}^d_+)$ and $G^\beta(\mathbb{R}^d_+)$ as an inductive limit of the spaces $G_{\alpha,A}(\mathbb{R}^d_+)$ and $G^{\beta,A}(\mathbb{R}^d_+)$, respectively, with respect to A:

$$G_\alpha(\mathbb{R}^d_+)=\varinjlim_{A\to\infty}G_{\alpha,A}(\mathbb{R}^d_+)\qquad\text{and}\qquad G^\beta(\mathbb{R}^d_+)=\varinjlim_{A\to\infty}G^{\beta,A}(\mathbb{R}^d_+).$$

Thus, $G_\alpha(\mathbb{R}^d_+)$ and $G^\beta(\mathbb{R}^d_+)$ are barrelled and bornological l.c.s. that are continuously injected into $\mathcal{S}(\mathbb{R}^d_+)$.

Remark 3.1.1. We will give an alternative definition (again as an inductive limit) of $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ which will be needed for the proof of the second structural theorem (see Subsection 3.7.2).

For $A > 0$, we denote by $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ the space of all $f \in \mathcal{S}(\mathbb{R}^d_+)$ such that

$$\sum_{p,k\in\mathbb{N}^d_0} \frac{\|x^{(p+k)/2}D^p f(x)\|^2_{L^2(\mathbb{R}^d_+)}}{A^{2|p+k|}k^{\alpha k}p^{\alpha p}} < \infty.$$

By (2.1), the space $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ with the seminorms

$$\tilde{\sigma}_{A,j}(f) = \left(\sum_{p,k\in\mathbb{N}^d_0} \frac{\|x^{(p+k)/2}D^p f(x)\|^2_{L^2(\mathbb{R}^d_+)}}{A^{2|p+k|}k^{\alpha k}p^{\alpha p}} + \sum_{|m|\leq j,\,|n|\leq j} \|x^m D^n f(x)\|^2_{L^2(\mathbb{R}^d_+)}\right)^{1/2},$$

$j \in \mathbb{N}_0$, becomes an (F)-space. When $A_1 < A_2$, $\tilde{G}^{\alpha,A_1}_{\alpha,A_1}(\mathbb{R}^d_+)$ is continuously injected into $\tilde{G}^{\alpha,A_2}_{\alpha,A_2}(\mathbb{R}^d_+)$. Clearly, $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ is continuously injected into $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ and $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ is continuously injected into $\tilde{G}^{\alpha,2A}_{\alpha,2A}(\mathbb{R}^d_+)$. So, $G^{\alpha}_{\alpha}(\mathbb{R}^d_+) = \varinjlim_{A\to\infty} \tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ as a l.c.s.

For each $m \in \mathbb{N}^d_0$, $f(t) \mapsto t^m f(t)$ is a continuous mapping $G_{\alpha}(\mathbb{R}^d_+) \to G_{\alpha}(\mathbb{R}^d_+)$, $G^{\beta}(\mathbb{R}^d_+) \to G^{\beta}(\mathbb{R}^d_+)$ and $G^{\beta}_{\alpha}(\mathbb{R}^d_+) \to G^{\beta}_{\alpha}(\mathbb{R}^d_+)$.

We denote by $(G^{\beta}(\mathbb{R}^d_+))'$, $(G_{\alpha}(\mathbb{R}^d_+))'$ and $(G^{\beta}_{\alpha}(\mathbb{R}^d_+))'$ the strong duals of $G^{\beta}(\mathbb{R}^d_+)$, $G_{\alpha}(\mathbb{R}^d_+)$ and $G^{\beta}_{\alpha}(\mathbb{R}^d_+)$, respectively.

One easily verifies that when $\alpha, \beta \geq 1$, $\mathcal{L}_n \in G^{\beta}_{\alpha}(\mathbb{R}^d_+)$ and hence $G^{\beta}_{\alpha}(\mathbb{R}^d_+)$ is dense in $\mathcal{S}(\mathbb{R}^d_+)$. In particular, for $\alpha \geq 1$, $G_{\alpha}(\mathbb{R}^d_+)$, $G^{\alpha}(\mathbb{R}^d_+)$ and $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ are dense in $\mathcal{S}(\mathbb{R}^d_+)$. Hence, $(\mathcal{S}(\mathbb{R}^d_+))'$ is continuously injected into $(G_{\alpha}(\mathbb{R}^d_+))'$, $(G^{\alpha}(\mathbb{R}^d_+))'$ and $(G^{\alpha}_{\alpha}(\mathbb{R}^d_+))'$.

Remark 3.1.2. Let $\alpha, \beta > 0$. Then the spaces $G^{\beta}_{\alpha}(\mathbb{R}^d_+)$ are non-trivial when $\alpha + \beta \geq 2$. We refer to Corollary 1.7.1 for d=1. For d-dimensional case it follows considering the function $\varphi(t) = \varphi_1(t_1)\dots\varphi_d(t_d)$, where φ_j, $j = 1, \dots, d$, is a non-zero element of $G^{\beta}_{\alpha}(\mathbb{R}_+)$.

3.2 The Hankel-Clifford transform

Let $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+})$ be a (B)-space of all continuous functions

$$f : \overline{\mathbb{R}^d_+} \to \mathbb{C} \text{ such that } \sup_{x\in\overline{\mathbb{R}^d_+}} |f(x)| < \infty,$$

the norm of $f \in \mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+})$ is given by the left-hand side.

For $\gamma \geq 0$, we denote by J_{γ} and I_{γ} the Bessel function of the first kind and the modified Bessel function of the first kind, respectively. Denote

$$\mathbf{T}^{(d)} = \{z \in \mathbb{C}^d \,|\, |z_l| = 1,\ z_l \neq 1,\ \forall l = 1, \dots, d\}.$$

For $z \in \mathbf{T}^{(d)}$ and $\gamma \in \overline{\mathbb{R}^d_+}$, we define the fractional powers and the modified fractional powers of the Hankel-Clifford transform of $f \in \mathcal{S}(\mathbb{R}^d_+)$ by

$$\begin{aligned}\mathcal{I}_{z,\gamma}f(t) &= \left(\prod_{l=1}^{d}(1-z_l)^{-1}e^{-\frac{1}{2}\frac{1+z_l}{1-z_l}t_l}\right)\int_{\mathbb{R}^d_+} f(x)\prod_{l=1}^{d} e^{-\frac{1}{2}\frac{1+z_l}{1-z_l}x_l}(x_l t_l z_l)^{-\gamma_l/2}x_l^{\gamma_l}\\ &\qquad \times I_{\gamma_l}\left(\frac{2\sqrt{x_l t_l z_l}}{1-z_l}\right)dx\\ \mathcal{J}_{z,\gamma}f(t) &= \left(\prod_{l=1}^{d}(1-z_l)^{-1}\right)\int_{\mathbb{R}^d_+} f(x)\prod_{l=1}^{d}(x_l t_l z_l)^{-\gamma_l/2}x_l^{\gamma_l}I_{\gamma_l}\left(\frac{2\sqrt{x_l t_l z_l}}{1-z_l}\right)dx.\end{aligned}$$

Since $z \in \mathbf{T}^{(d)}$,

$$z_l = e^{i\theta_l} \text{ where } \theta_l \in (-\pi,\pi]\backslash\{0\},\ l = 1,\ldots,d.$$

Observe that $(1+z_l)/(1-z_l)$ is purely imaginary. Moreover,

$$\frac{2\sqrt{x_l t_l z_l}}{(1-z_l)} = \frac{i\sqrt{x_l t_l}}{\sin(\theta_l/2)} \quad \text{and} \quad (x_l t_l z_l)^{-\frac{\gamma_l}{2}} = (x_l t_l)^{-\frac{\gamma_l}{2}}e^{-\frac{i\theta_l\gamma_l}{2}}.$$

Hence, for $l = 1,\ldots,d$,

$$\begin{aligned}(x_l t_l z_l)^{-\gamma_l/2}I_{\gamma_l}\left(\frac{2\sqrt{x_l t_l z_l}}{1-z_l}\right) &= e^{-i\theta_l\gamma_l/2}(x_l t_l)^{-\gamma_l/2}e^{(i\gamma_l\pi \operatorname{sgn}\theta_l)/2}\\ &\qquad \times J_{\gamma_l}\left(\frac{\sqrt{x_l t_l}}{|\sin(\theta_l/2)|}\right).\end{aligned} \tag{3.1}$$

By the definition of the Bessel function of the first kind, it is clear that for $\nu \geq 0$, $\xi^{-\nu}|J_\nu(\xi)|$ is uniformly bounded when $\xi \in (0,c)$ for arbitrary but fixed $c \geq 1$. Combining this with the asymptotic expansion of the Bessel function of the first kind (see [1, 9.2.1, p. 364])

$$J_\nu(\xi) = \sqrt{2/(\pi\xi)}\{\cos(\xi - \frac{1}{2}\nu\pi - \frac{1}{4}\pi) + e^{|\Im z|}\mathcal{O}(|z|^{-1})\}, |z| \to \infty, |\arg z| < \pi,$$

we obtain that there exists $C \geq 1$ such that

$$\left|\prod_{l=1}^{d}(x_l t_l z_l)^{-\gamma_l/2}I_{\gamma_l}\left(\frac{2\sqrt{x_l t_l z_l}}{1-z_l}\right)\right| \leq C,\ \forall x,t \in \mathbb{R}^d_+. \tag{3.2}$$

Moreover, for $\nu \geq 0$, by the definition of J_ν, the function

$$\xi \mapsto \xi^{-\nu}J_\nu(\xi),\ \mathbb{R}_+ \to \mathbb{C},$$

can be extended to a continuous function on $\overline{\mathbb{R}_+}$. Hence, (3.1) and (3.2) imply that for $f \in \mathcal{S}(\mathbb{R}^d_+)$ the integrals in the definition for $\mathcal{I}_{z,\gamma}f$ and $\mathcal{J}_{z,\gamma}f$ converge absolutely i.e.

$$\mathcal{I}_{z,\gamma}f, \mathcal{J}_{z,\gamma}f \in \mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+}).$$

When $f_j \to f$ in $\mathcal{S}(\mathbb{R}^d_+)$,

$$\mathcal{I}_{z,\gamma} f_j \to \mathcal{I}_{z,\gamma} f \text{ and } \mathcal{J}_{z,\gamma} f_j \to \mathcal{J}_{z,\gamma} f \text{ in } \mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+}).$$

Hence, $\mathcal{I}_{z,\gamma}$ and $\mathcal{J}_{z,\gamma}$ are well defined continuous mappings from $\mathcal{S}(\mathbb{R}^d_+)$ to $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+})$. Our goal is to prove that $\mathcal{I}_{z,\gamma}$ and $\mathcal{J}_{z,\gamma}$ are continuous mappings from $\mathcal{S}(\mathbb{R}^d_+)$ onto $\mathcal{S}(\mathbb{R}^d_+)$. Firstly, we prove this for $\mathcal{J}_{z,\gamma}$ in the case $d = 1$.

Lemma 3.2.1. *For $z \in \mathbf{T}^{(1)}$ and $\gamma \geq 0$, $\mathcal{J}_{z,\gamma}$ is a continuous mapping from $\mathcal{S}(\mathbb{R}_+)$ onto $\mathcal{S}(\mathbb{R}_+)$.*

Proof. Clearly $\mathcal{S}(\mathbb{R}_+)$ is continuously injected into $L^2(\mathbb{R}_+, t^\gamma dt)$. Let E_γ be the Laguerre operator. Then

$$E_\gamma(t^{\gamma/2}\mathcal{L}^\gamma_n(t)) = -nt^{\gamma/2}\mathcal{L}^\gamma_n(t),$$

see (1.3). For $f \in \mathcal{S}(\mathbb{R}_+)$, we have

$$E_\gamma(t^{\gamma/2}f(t)) = t^{\gamma/2}\Big((\gamma+1)Df(t) + tD^2f(t) - \frac{tf(t)}{4} + \frac{(\gamma+1)}{2}f(t)\Big).$$

Hence, for $k \in \mathbb{N}$,

$$E^k_\gamma(t^{\gamma/2}f(t)) = t^{\gamma/2}g_k(t),$$

for some $g_k \in \mathcal{S}(\mathbb{R}_+)$. Let

$$a_n(f) = \int_0^\infty f(t)\mathcal{L}^\gamma_n(t)t^\gamma dt.$$

Then, by integration by parts, we have

$$\int_0^\infty g_1(t)\mathcal{L}^\gamma_n(t)t^\gamma dt = \int_0^\infty E_\gamma(t^{\gamma/2}f(t))\mathcal{L}^\gamma_n(t)t^{\gamma/2}dt = -na_n(f).$$

Iterating this, we obtain

$$\int_0^\infty g_k(t)\mathcal{L}^\gamma_n(t)t^\gamma dt = (-n)^k a_n(f).$$

Since $g_k \in \mathcal{S}(\mathbb{R}_+) \subseteq L^2(\mathbb{R}_+, t^\gamma dt)$, we conclude $\{a_n(f)\}_{n\in\mathbb{N}_0} \in s$. Observe that $f = \sum_n a_n(f)\mathcal{L}^\gamma_n$ in $L^2(\mathbb{R}_+, t^\gamma dt)$. We need the following estimate for the derivatives of the Laguerre polynomials (see Theorem 1.4.1):

$$\left|t^kD^p(e^{-t/2}L^\gamma_n(t))\right| \leq 2^{-\min\{\gamma,k\}}4^k(n+1)\cdot\ldots\cdot(n+k)\binom{n+\max\{\gamma-k,0\}+p}{n},$$

for all $t \geq 0$, $n, k, p \in \mathbb{N}_0$. Denote by $[\gamma]$ the integral part of γ, we have

$$\binom{n+\max\{\gamma-k,0\}+p}{n} \leq \binom{n+[\gamma]+1+p}{n} \leq (n+[\gamma]+p+1)^{[\gamma]+p+1}.$$

Hence, there exists $C_{p,k} \geq 1$ which depends on p and k, but not on n, such that

$$\left|t^k D^p \mathcal{L}_n^\gamma(t)\right| \leq C_{p,k}(n+1)^{k+p+[\gamma]+1}. \tag{3.3}$$

Since $\{a_n(f)\}_n \in s$, we have

$$\sum_n |a_n(f)| \sup_{t \in \mathbb{R}_+} |t^k D^p \mathcal{L}_n^\gamma(t)| < \infty,$$

i.e. $\sum_n a_n(f)\mathcal{L}_n^\gamma$ converges absolutely in $\mathcal{S}(\mathbb{R}_+)$. Since $\mathcal{J}_{z,\gamma} f : \mathcal{S}(\mathbb{R}_+) \to \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$ is continuous,

$$\mathcal{J}_{z,\gamma} f = \sum_n a_n(f) \mathcal{J}_{z,\gamma} \mathcal{L}_n^\gamma$$

and the series converges absolutely in $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$. Using (3.1) and (1.9), we obtain

$$\mathcal{J}_{z,\gamma}\mathcal{L}_n^\gamma(t) = 2(-1)^n e^{-\frac{i\gamma\theta}{2}} e^{\frac{i\gamma\pi \operatorname{sgn}\theta}{2}} (1-e^{i\theta})^{-1} |\sin(\theta/2)|^{-\gamma} \mathcal{L}_n^\gamma\left(t/\sin^2(\theta/2)\right). \tag{3.4}$$

The estimate (3.3) together with (3.4) implies that $\sum_n a_n(f)\mathcal{J}_{z,\gamma}\mathcal{L}_n^\gamma$ converges absolutely in $\mathcal{S}(\mathbb{R}_+)$. Thus, we obtain that the image of $\mathcal{S}(\mathbb{R}_+)$ under $\mathcal{J}_{z,\gamma}$ is contained in $\mathcal{S}(\mathbb{R}_+)$. Since $\mathcal{J}_{z,\gamma} : \mathcal{S}(\mathbb{R}_+) \to \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$ is continuous its graph is closed in $\mathcal{S}(\mathbb{R}_+) \times \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$. As $\mathcal{S}(\mathbb{R}_+)$ is continuously injected into $\mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+})$ and $\mathcal{J}_{z,\gamma}(\mathcal{S}(\mathbb{R}_+)) \subseteq \mathcal{S}(\mathbb{R}_+)$, the graph of $\mathcal{J}_{z,\gamma}$ is closed in $\mathcal{S}(\mathbb{R}_+) \times \mathcal{S}(\mathbb{R}_+)$. Since $\mathcal{S}(\mathbb{R}_+)$ is an (F)-space, the closed graph theorem implies that $\mathcal{J}_{z,\gamma} : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+)$ is continuous (see Appendix A.3). $\square$

Now, by the principle of induction, we show that for $z \in \mathbf{T}^{(d)}$ and $\gamma \in \overline{\mathbb{R}_+^d}$, $\mathcal{J}_{z,\gamma}$ is a continuous mapping from $\mathcal{S}(\mathbb{R}_+^d)$ into itself. When $f \in \mathcal{S}(\mathbb{R}_+^d)$, we denote $\mathcal{J}_{z,\gamma}$ by $\mathcal{J}_{z,\gamma}^{(d)}$ in order to avoid confusions. We already considered the case $d=1$; $\mathcal{J}_{z,\gamma}^{(1)} : \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+)$ is continuous. Let $\mathcal{J}_{z,\gamma}^{(d)}$ be continuous. Let

$$\nu = (\gamma, \gamma') \in \overline{\mathbb{R}_+^{d+1}} \text{ where } \gamma \in \overline{\mathbb{R}_+^d} \text{ and } \gamma' \geq 0$$

and let

$$\zeta = (z, z') \in \mathbf{T}^{(d+1)} \text{ where } z \in \mathbf{T}^{(d)} \text{ and } z' \in \mathbf{T}^{(1)}.$$

The mapping

$$\mathcal{J}_{z,\gamma}^{(d)} \otimes \mathcal{J}_{z',\gamma'}^{(1)} : \mathcal{S}(\mathbb{R}_+^d) \otimes_\pi \mathcal{S}(\mathbb{R}_+) \to \mathcal{S}(\mathbb{R}_+^d) \otimes_\pi \mathcal{S}(\mathbb{R}_+)$$

is continuous. Denoting by $\tilde{\mathcal{J}}_{\zeta,\nu}$ its continuous extension on the completions, the Schwartz kernel theorem i.e. Theorem 2.4.1 for $\mathcal{S}(\mathbb{R}_+^d)$ yields that $\tilde{\mathcal{J}}_{\zeta,\nu}$ is a continuous mapping from $\mathcal{S}(\mathbb{R}_+^{d+1})$ into itself. Observe that for each $f \in \mathcal{S}(\mathbb{R}_+^d) \otimes \mathcal{S}(\mathbb{R}_+)$,

$$\mathcal{J}_{\zeta,\nu}^{(d+1)} f(t) = \tilde{\mathcal{J}}_{\zeta,\nu} f(t), \ \forall t \in \mathbb{R}^{d+1}.$$

Thus $\mathcal{J}_{\zeta,\nu}^{(d+1)} f \in \mathcal{S}(\mathbb{R}_+^{d+1})$. If $f \in \mathcal{S}(\mathbb{R}_+^{d+1})$, there exists a sequence

$$f_j \in \mathcal{S}(\mathbb{R}_+^d) \otimes \mathcal{S}(\mathbb{R}_+), \ j \in \mathbb{N}, \qquad \text{such that } f_j \to f \text{ in } \mathcal{S}(\mathbb{R}_+^{d+1})$$

(cf. Theorem 2.4.1; $\mathcal{S}(\mathbb{R}_+^{d+1})$ is an (F)-space). Since we proved that $\mathcal{J}_{\zeta,\nu}^{(d+1)}$: $\mathcal{S}(\mathbb{R}_+^{d+1}) \to \mathcal{C}_{L^\infty}(\overline{\mathbb{R}_+^{d+1}})$ is continuous (see the discussion before Lemma 3.2.1), we have, for each fixed $t \in \mathbb{R}_+^{d+1}$,

$$\mathcal{J}_{\zeta,\nu}^{(d+1)} f(t) = \lim_{j\to\infty} \mathcal{J}_{\zeta,\nu}^{(d+1)} f_j(t) = \lim_{j\to\infty} \tilde{\mathcal{J}}_{\zeta,\nu} f_j(t) = \tilde{\mathcal{J}}_{\zeta,\nu} f(t).$$

Hence,

$$\mathcal{J}_{\zeta,\nu}^{(d+1)} f \in \mathcal{S}(\mathbb{R}_+^{d+1}) \qquad \text{and} \qquad \mathcal{J}_{\zeta,\nu}^{(d+1)} f = \tilde{\mathcal{J}}_{\zeta,\nu} f, \quad \forall f \in \mathcal{S}(\mathbb{R}_+^{d+1}).$$

We conclude that $\mathcal{J}_{\zeta,\nu}^{(d+1)} : \mathcal{S}(\mathbb{R}_+^{d+1}) \to \mathcal{S}(\mathbb{R}_+^{d+1})$ is continuous.

Next we prove that $\mathcal{J}_{z,\gamma}$ extends to isometry from $L^2(\mathbb{R}_+^d, t^\gamma dt)$ onto itself. Firstly, we prove the following claim:

For $\gamma \in \overline{\mathbb{R}_+^d}$, let $V_\gamma^{(d)} \subseteq \mathcal{S}(\mathbb{R}_+^d)$ be the space which consists of all finite linear combinations of the form

$$\sum_{k\leq n} a_k \mathcal{L}_k^\gamma, \qquad a_k \in \mathbb{C}.$$

Then, for each $\gamma \in \overline{\mathbb{R}_+^d}$, $V_\gamma^{(d)}$ is dense in $\mathcal{S}(\mathbb{R}_+^d)$.

The proof follows by the principle of induction on the dimension. For $d = 1$, it is already proved in the first part of the proof of Lemma 3.2.1. Assume that the assertion holds for $d \in \mathbb{N}$. Let $\nu = (\gamma, \gamma') \in \overline{\mathbb{R}_+^{d+1}}$, where $\gamma \in \overline{\mathbb{R}_+^d}$ and $\gamma' \geq 0$. The inductive hypothesis implies that

$$V_\gamma^{(d)} \otimes V_{\gamma'}^{(1)} \text{ is dense in } \mathcal{S}(\mathbb{R}_+^d) \otimes_\epsilon \mathcal{S}(\mathbb{R}_+)$$

and consequently in $\mathcal{S}(\mathbb{R}_+^{d+1})$ by the Schwartz kernel theorem for $\mathcal{S}(\mathbb{R}_+^d)$, i.e. Theorem 2.4.1. One easily verifies that $V_\gamma^{(d)} \otimes V_{\gamma'}^{(1)} \subseteq V_\nu^{(d+1)}$ and the proof is completed.

By (3.1) and (1.9), we obtain

$$\begin{aligned} \mathcal{J}_{z,\gamma} \mathcal{L}_n^\gamma(t) &= 2^d (-1)^{|n|} c_{z,\gamma} \left(\prod_{l=1}^d |\sin(\theta_l/2)|^{-\gamma_l} \right) \\ &\quad \times \mathcal{L}_n^\gamma \left(\frac{t_1}{\sin^2(\theta_1/2)}, \ldots, \frac{t_d}{\sin^2(\theta_d/2)} \right), \end{aligned} \tag{3.5}$$

where

$$c_{z,\gamma} = \prod_{l=1}^d e^{-\frac{i\gamma_l \theta_l}{2}} e^{\frac{i\gamma_l \pi \operatorname{sgn} \theta_l}{2}} (1 - e^{i\theta_l})^{-1}.$$

One easily verifies that the set $\{\mathcal{J}_{z,\gamma} \mathcal{L}_n^\gamma | \, n \in \mathbb{N}_0^d\}$ is orthonormal in $L^2(\mathbb{R}_+^d, t^\gamma dt)$. Now, we have

$$\|\mathcal{J}_{z,\gamma} f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)} = \|f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)} \text{ for } f \in V_\gamma^{(d)}$$

($V_\gamma^{(d)}$ is a subspace of $\mathcal{S}(\mathbb{R}_+^d)$ defined in the assertion above). Since $V_\gamma^{(d)}$ is dense in $\mathcal{S}(\mathbb{R}_+^d)$, we have

$$\|\mathcal{J}_{z,\gamma} f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)} = \|f\|_{L^2(\mathbb{R}_+^d, t^\gamma dt)} \text{ for all } f \in \mathcal{S}(\mathbb{R}_+^d). \tag{3.6}$$

Thus $\mathcal{J}_{z,\gamma}$ extends to an isometry from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ into itself. Secondly, we prove the surjectivity of $\mathcal{J}_{z,\gamma}$. It follows from (1.9) and (3.5) that

$$\mathcal{J}_{\bar{z},\gamma}\mathcal{J}_{z,\gamma}\mathcal{L}^\gamma_n = \mathcal{L}^\gamma_n \text{ and } \mathcal{J}_{z,\gamma}\mathcal{J}_{\bar{z},\gamma}\mathcal{L}^\gamma_n = \mathcal{L}^\gamma_n, \text{ where } \bar{z} = (\bar{z}_1, \dots, \bar{z}_d).$$

Hence, $\mathcal{J}_{z,\gamma} : L^2(\mathbb{R}^d_+, t^\gamma dt) \to L^2(\mathbb{R}^d_+, t^\gamma dt)$ is bijective with an inverse $\mathcal{J}_{\bar{z},\gamma}$. Incidentally, we can also conclude that $\mathcal{J}_{z,\gamma} : \mathcal{S}(\mathbb{R}^d_+) \to \mathcal{S}(\mathbb{R}^d_+)$ is a topological isomorphism (has an inverse $\mathcal{J}_{\bar{z},\gamma}$).

Let $z \in \mathbf{T}^{(d)}$ and

$$\Phi_z(t) = \prod_{l=1}^d e^{-\frac{1}{2}\frac{1+z_l}{1-z_l}t_l}.$$

Since $(1+z_l)/(1-z_l)$ is purely imaginary, for all $l = 1, \dots, d$, $|\Phi_z(t)| = 1$ and one easily verifies that the mapping $f \mapsto \Phi_z f$, is a topological isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$ and an isometry from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ onto itself. Since

$$\mathcal{I}_{z,\gamma} f = \Phi_z \mathcal{J}_{z,\gamma}(\Phi_z f),$$

we can conclude that $\mathcal{I}_{z,\gamma}$ is a topological isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$ and isometry from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ onto itself; clearly, its inverse is $\mathcal{I}_{\bar{z},\gamma}$. In the sequel, we will need this technical lemma.

Lemma 3.2.2. *([8, Lemma 3.2] for $d = 1$) Let $f \in \mathcal{S}(\mathbb{R}^d_+)$, $\gamma \in \overline{\mathbb{R}^d_+}$, $z \in \mathbb{C}^d$, $|z| = 1$, $z \neq 1$ and $n \in \mathbb{N}^d_0$.*

(i) If $0 \le k \le n$ then

$$\mathcal{J}_{z,\gamma+n} f(t) = \prod_{l=1}^d \Big(\frac{1-z_l}{z_l}\Big)^{n_l-k_l} D^{n-k}\mathcal{J}_{z,\gamma+k} f(t).$$

(ii) $\mathcal{J}_{z,\gamma} f(t) = \prod_{l=1}^d (z_l - 1)^{n_l} D^n \mathcal{J}_{z,\gamma+n} f(t)$.

(iii)

$$\big\|t^{(p+k+\gamma)/2} D^p f(t)\big\|_2 = \left(\prod_{l=1}^d |1-z_l|^{-p_l+k_l}\right) \big\|t^{(p+k+\gamma)/2} D^k \mathcal{J}_{z,\gamma} f(t)\big\|_2, \quad (3.7)$$

for $p, k \in \mathbb{N}^d_0$.

Proof. We follow the proof of (1.17).

(i) Since

$$(t^{-\frac{\gamma}{2}} I_\gamma(\sqrt{t}))^{(m)} = \frac{1}{2^m} t^{\frac{\gamma+m}{2}} I_{\gamma+m}(\sqrt{t})$$

(see [25, p.103]), we obtain

$$\mathcal{J}_{z,\gamma+k} f(t) = \Big(\prod_{l=1}^d \Big(\frac{z_l}{1-z_l}\Big)^{n_l-k_l}\Big)$$

$$\times \int_{\mathbb{R}_+^d} f(x) \prod_{l=1}^{d} (x_l t_l z_l)^{-(\gamma_l+n_l)/2} x_l^{\gamma_l+n_l} I_{\gamma_l+n_l}\left(\frac{2\sqrt{x_l t_l z_l}}{1-z_l}\right) dx$$

$$= \Big(\prod_{l=1}^{d} \Big(\frac{z_l}{1-z_l}\Big)^{n_l-k_l}\Big) \mathcal{J}_{z,\gamma+n} f(t).$$

(ii) Since, $f = \mathcal{J}_{z,\gamma}^{-1}\mathcal{J}_{z,\gamma} f(t) = \mathcal{J}_{z^{-1},\gamma}(\mathcal{J}_{z,\gamma} f(t))$, we obtain

$$D^n f(t) = D^n \mathcal{J}_{z^{-1},\gamma}(\mathcal{J}_{z,\gamma} f(t)).$$

Hence, from (i) follows

$$D^n f(t) = \Big(\prod_{l=1}^{d} \Big(\frac{1}{z_l-1}\Big)^{n_l}\Big) \mathcal{J}_{z^{-1},\gamma+n}(\mathcal{J}_{z,\gamma} f(t)).$$

Hence, the assertion follows.

(iii) Consider $F \in \mathcal{S}(\mathbb{R}_+^d)$ such that $D^k F = f$ and put $g = \mathcal{J}_{z,\gamma} F \in \mathcal{S}(\mathbb{R}_+^d)$. Then from (i) and (ii) follows

$$\begin{aligned} D^n f &= D^{n+k} F = D^{n+k} \mathcal{J}_{z,\gamma}^{-1} g = D^{n+k} \mathcal{J}_{z^{-1},\gamma} g \\ &= \Big(\prod_{l=1}^{d} \Big(\frac{1}{z_l-1}\Big)^{n_l+k_l}\Big) \mathcal{J}_{z^{-1},\gamma+n+k} g \end{aligned}$$

and

$$\begin{aligned} g &= \mathcal{J}_{z,\gamma} f = \Big(\prod_{l=1}^{d} (z_l-1)^{k_l}\Big) \mathcal{J}_{z,\gamma+k} D^k F = \Big(\prod_{l=1}^{d} (z_l-1)^{k_l}\Big) \mathcal{J}_{z,\gamma+k} f \\ &= \Big(\prod_{l=1}^{d} \Big(\frac{(1-z_l)(z_l-1)}{z_l}\Big)^{k_l}\Big) D^k \mathcal{J}_{z,\gamma} f. \end{aligned}$$

Now, from (3.6), we obtain

$$\begin{aligned} \left\| t^{(p+k+\gamma)/2} D^p f(t) \right\|_2 &= \left(\prod_{l=1}^{d} |1-z_l|^{-p_l-k_l}\right) \left\| t^{(p+k+\gamma)/2} \mathcal{J}_{z^{-1},\gamma+n+k} g(t) \right\|_2 \\ &= \left(\prod_{l=1}^{d} |1-z_l|^{-p_l-k_l}\right) \left\| t^{(p+k+\gamma)/2} g(t) \right\|_2 \\ &= \left(\prod_{l=1}^{d} |1-z_l|^{-p_l+k_l}\right) \left\| t^{(p+k+\gamma)/2} D^k \mathcal{J}_{z,\gamma} f(t) \right\|_2 . \end{aligned}$$

□

Next, we summarise the properties of $\mathcal{J}_{z,\gamma}$ and $\mathcal{I}_{z,\gamma}$ in the following proposition:

Proposition 3.2.1. *For $\gamma \in \overline{\mathbb{R}^d_+}$ and $z \in \mathbf{T}^{(d)}$ the fractional powers and the modified fractional powers of the Hankel-Clifford transform $\mathcal{I}_{z,\gamma}$ and $\mathcal{J}_{z,\gamma}$ are topological isomorphisms on $\mathcal{S}(\mathbb{R}^d_+)$ and they extend to isometries from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ onto itself with inverses, $\mathcal{I}_{\bar{z},\gamma}$ and $\mathcal{J}_{\bar{z},\gamma}$ respectively. Moreover, for all $p, k \in \mathbb{N}^d_0$ and $f \in \mathcal{S}(\mathbb{R}^d_+)$, (3.7) is valid.*

Notice that when $z = -\mathbf{1} \in \mathbf{T}^{(d)}$ then $\mathcal{H}_\gamma = \mathcal{J}_{z,\gamma} = \mathcal{I}_{z,\gamma}$ where $\mathcal{H}_\gamma$ is the d-dimensional Hankel-Clifford transform, defined by

$$\mathcal{H}_\gamma(f)(t) = 2^{-d} t^{-\gamma/2} \int_{\mathbb{R}^d_+} f(x) x^{\gamma/2} \prod_{l=1}^{d} J_{\gamma_l}(\sqrt{x_l t_l}) dx, \ t \in \mathbb{R}^d_+.$$

By (3.5), $\mathcal{L}^\gamma_n$, $n \in \mathbb{N}^d_0$, are eigenfunctions for $\mathcal{H}_\gamma$; more precisely

$$\mathcal{H}_\gamma \mathcal{L}^\gamma_n = (-1)^{|n|} \mathcal{L}^\gamma_n.$$

Since $\mathcal{J}_{z,0}$ is an isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$, by (3.7) we have the following result.

Theorem 3.2.1. *The modified fractional powers of the Hankel-Clifford transform $\mathcal{J}_{z,0}$ are isomorphisms of $G_\alpha(\mathbb{R}^d_+)$, $G^\beta(\mathbb{R}^d_+)$ and $G^\beta_\alpha(\mathbb{R}^d_+)$ onto $G^\alpha(\mathbb{R}^d_+)$, $G_\beta(\mathbb{R}^d_+)$ and $G^\alpha_\beta(\mathbb{R}^d_+)$ respectively.*

Let

- $d', d'' \in \mathbb{N}$,
- $\gamma = (\gamma', \gamma'') \in \overline{\mathbb{R}^{d'}_+} \times \overline{\mathbb{R}^{d''}_+} = \overline{\mathbb{R}^d_+}$ (for brevity $d = d' + d''$) and
- $z' = (z_1, \ldots, z_{d'}) \in \mathbf{T}^{(d')}$.

Denote by $\mathcal{J}^{d'}_{z',\gamma'}$ the modified fractional power of the Hankel-Clifford transform on $\mathbb{R}^{d'}_+$ and by $\mathrm{Id}^{d''}$ the identity operator $\mathcal{S}(\mathbb{R}^{d''}_+) \to \mathcal{S}(\mathbb{R}^{d''}_+)$. Now, since:

Theorem 3.2.2. *([23, Theorem 5, p. 277]) Let E_1, E_2, F_1, F_2 be locally convex, $A_1 \in \mathcal{L}(E_1, F_1), A_2 \in \mathcal{L}(E_2, F_2)$. If A_1, A_2 are injections, then $A_1 \otimes_\varepsilon A_2$ is an injection. If E_1, E_2, F_1, F_2 are complete, then also $A_1 \hat{\otimes}_\varepsilon A_2$ is an injection.*

Note, $\mathcal{L}(E, F)$ is the vector space of all continuous linear mappings of E into F.

It follows that $\mathcal{J}^{d'}_{z',\gamma'} \hat{\otimes} \mathrm{Id}^{d''}$ is an injection on $\mathcal{S}(\mathbb{R}^d_+)$. From

Theorem 3.2.3. *([23, Theorem 7, p. 189]) If E_1, E_2 are metrizable and A_1, A_2 homomorphisms onto F_1 and F_2, respectively, then $A_1 \hat{\otimes}_\pi A_2$ is a homomorphism onto $F_1 \hat{\otimes}_\pi F_2$.*

follows that $\mathcal{J}^{d'}_{z',\gamma'} \hat{\otimes} \mathrm{Id}^{d''}$ is a homomorphism on $\mathcal{S}(\mathbb{R}^d_+)$. Now, since $\mathcal{S}(\mathbb{R}^d_+)$ is nuclear (see Theorem 2.4.1), Proposition 3.2.1 imply that $\mathcal{J}^{d'}_{z',\gamma'} \hat{\otimes} \mathrm{Id}^{d''}$ is a topological isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$. We denote by $x \in \mathbb{R}^d_+$ $x = (x', x'')$, where $x' = (x_1, \ldots, x_{d'})$

and $x'' = (x_{d'+1}, \dots, x_d)$. Let $f \in \mathcal{S}(\mathbb{R}^d_+)$. Define the modified fractional power of the partial Hankel-Clifford transform

$$\mathcal{J}^{(d')}_{z',\gamma'} f(t) = \left(\prod_{l=1}^{d'} (1-z_l)^{-1}\right) \int_{\mathbb{R}^{d'}_+} f(x', t'') \prod_{l=1}^{d'} (x_l t_l z_l)^{-\gamma_l/2} x_l^{\gamma_l} I_{\gamma_l}\left(\frac{2\sqrt{x_l t_l z_l}}{1-z_l}\right) dx'.$$

By the same technique already described for the absolute convergence of $\mathcal{J}_{z,\gamma}$, one proves that $\mathcal{J}^{(d')}_{z',\gamma'} f \in \mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+})$. When $f_j \to f$ in $\mathcal{S}(\mathbb{R}^d_+)$,

$$\mathcal{J}^{(d')}_{z',\gamma'} f_j \to \mathcal{J}^{(d')}_{z',\gamma'} f \text{in } \mathcal{C}_{L^\infty}(\overline{\mathbb{R}^d_+}).$$

Since,

$$\mathcal{J}^{(d')}_{z',\gamma'} f(t) = \mathcal{J}^{d'}_{z',\gamma'} \hat{\otimes} \mathrm{Id}^{d''} f(t) \text{ for } f \in \mathcal{S}(\mathbb{R}^{d'}_+) \otimes \mathcal{S}(\mathbb{R}^{d''}_+),$$

we accomplish the same for all $f \in \mathcal{S}(\mathbb{R}^d_+)$. Hence, the first part of the next proposition follows.

Proposition 3.2.2. *The modified fractional power of the partial Hankel-Clifford transform $\mathcal{J}^{(d')}_{z',\gamma'}$ is a topological isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$.*

Moreover, $\mathcal{J}^{(d')}_{z',\gamma'}$ extends to an isometry from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ onto itself with an inverse $\mathcal{J}^{(d')}_{\bar{z}',\gamma'}$. For all $(p', p''), (k', k'') \in \mathbb{N}^{d'}_0 \times \mathbb{N}^{d''}_0 = \mathbb{N}^d_0$ and all $f \in \mathcal{S}(\mathbb{R}^d_+)$

$$\begin{aligned}&\left\|t'^{(p'+k'+\gamma')/2} t''^{(p''+k'')/2} D^p_t f(t)\right\|_2 \\ &\quad = \left(\prod_{l=1}^{d'} |1-z_l|^{-p_l+k_l}\right) \left\|t'^{(p'+k'+\gamma')/2} t''^{(p''+k'')/2} D^{k'}_{t'} D^{p''}_{t''} \mathcal{J}^{(d')}_{z',\gamma'} f(t)\right\|_2 .\end{aligned}$$

Proof. The proof that $\mathcal{J}^{(d')}_{z',\gamma'}$ extends to an isometry from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ onto itself with an inverse $\mathcal{J}^{(d')}_{\bar{z}',\gamma'}$ is the same as for $\mathcal{J}_{z,\gamma}$ given above. As in the proof of (3.7), one obtains for $f \in \mathcal{S}(\mathbb{R}^d_+)$

$$\begin{aligned}&\left\|t'^{(p'+k'+\gamma')/2} t''^{(p''+k'')/2} D^{p'}_{t'} f(t)\right\|_2 \\ &\quad = \left(\prod_{l=1}^{d'} |1-z_l|^{-p_l+k_l}\right) \left\|t'^{(p'+k'+\gamma')/2} t''^{(p''+k'')/2} D^{k'}_{t'} \mathcal{J}^{(d')}_{z',\gamma'} f(t)\right\|_2 .\end{aligned} \tag{3.8}$$

Clearly,

$$D^{p''}_{t''} \mathcal{J}^{(d')}_{z',\gamma'} f = \mathcal{J}^{(d')}_{z',\gamma'} D^{p''}_{t''} f, \text{ for } f \in \mathcal{S}(\mathbb{R}^{d'}_+) \otimes \mathcal{S}(\mathbb{R}^{d''}_+).$$

Hence, the same holds for $f \in \mathcal{S}(\mathbb{R}^d_+)$ and the equality follows from (3.8). □

If $\Lambda' = \{\lambda'_1, \dots, \lambda'_{d'}\} \subseteq \{1, \dots, d\}$ and $\Lambda'' = \{\lambda''_1, \dots, \lambda''_{d''}\} = \{1, \dots, d\} \backslash \Lambda'$ one can also consider the modified fractional power of the partial Hankel-Clifford transform with respect to $x_{\Lambda'} = (x_{\lambda'_1}, \dots, x_{\lambda'_{d'}})$ defined by (here $x_{\Lambda''} = (x_{\lambda''_1}, \dots, x_{\lambda''_{d''}})$ and abusing the notation we write $x = (x_{\Lambda'}, x_{\Lambda''})$)

$$\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}} f(t) \quad = \quad \left(\prod_{l=1}^{d'} (1-z_l)^{-1}\right) \int_{\mathbb{R}^{d'}_+} f(x_{\Lambda'}, t_{\Lambda''}) \prod_{l=1}^{d'} (x_{\lambda'_l} t_{\lambda'_l} z_l)^{-\gamma_{\lambda'_l}/2} x_{\lambda'_l}^{\gamma_{\lambda'_l}}$$

$$\times I_{\gamma_{\lambda'_l}}\left(\frac{2\sqrt{x_{\lambda'_l}t_{\lambda'_l}z_l}}{1-z_l}\right)dx_{\Lambda'}.$$

Corollary 3.2.1. *Using the same notations as above, $\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}}$ is a topological isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$ and it extends to an isometry from $L^2(\mathbb{R}^d_+, t^\gamma dt)$ onto itself with an inverse $\mathcal{J}^{(\Lambda')}_{\bar{z}',\gamma_{\Lambda'}}$. For all $f\in\mathcal{S}(\mathbb{R}^d_+)$ and all $(p_{\Lambda'},p_{\Lambda''}),(k_{\Lambda'},k_{\Lambda''})\in\mathbb{N}^d_0$*

$$\left\|t_{\Lambda'}^{(p_{\Lambda'}+k_{\Lambda'}+\gamma_{\Lambda'})/2}t_{\Lambda''}^{(p_{\Lambda''}+k_{\Lambda''})/2}D^p_t f(t)\right\|_2$$

$$=\left(\prod_{l=1}^{d'}|1-z_l|^{-p_{\lambda'_l}+k_{\lambda'_l}}\right)\left\|t_{\Lambda'}^{(p_{\Lambda'}+k_{\Lambda'}+\gamma_{\Lambda'})/2}t_{\Lambda''}^{(p_{\Lambda''}+k_{\Lambda''})/2}D^{k_{\Lambda'}}_{t_{\Lambda'}}D^{p_{\Lambda''}}_{t_{\Lambda''}}\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}}f(t)\right\|_2. \quad (3.9)$$

Proof. Let

$$\Theta:\mathbb{R}^d\to\mathbb{R}^d$$

be the orthogonal transformation given by

$$\Theta(x)=y, \text{ where } y_{\lambda'_1}=x_1,\dots,y_{\lambda'_{d'}}=x_{d'} \text{ and } y_{\lambda''_1}=x_{d'+1},\dots,y_{\lambda''_{d''}}=x_{d''}.$$

Observe that Θ maps $\mathbb{R}^d_+$ and $\overline{\mathbb{R}^d_+}$ bijectively onto themselves. Let $\tilde{\Theta}$ be the mapping

$$f\mapsto f\circ\Theta,\ L^2(\mathbb{R}^d_+)\to L^2(\mathbb{R}^d_+).$$

One easily verifies that for each $\mu\in\overline{\mathbb{R}^d_+}$ it is an isometry from $L^2(\mathbb{R}^d_+,t^\mu dt)$ onto $L^2(\mathbb{R}^d_+,t^{\Theta^{-1}\mu}dt)$ and a topological isomorphism on $\mathcal{S}(\mathbb{R}^d_+)$. Its inverse is

$$\tilde{\Theta}^{-1}f=f\circ\Theta^{-1}.$$

Let $\nu'=(\gamma_{\lambda'_1},\dots,\gamma_{\lambda'_{d'}})\in\overline{\mathbb{R}^{d'}_+}$. The corollary follows from Proposition 3.2.2 and the fact that

$$\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}}f=\tilde{\Theta}^{-1}\mathcal{J}^{(d')}_{z',\nu'}\tilde{\Theta}f.$$

□

Remark 3.2.1. Observe that

- if $\Lambda'=\emptyset$ then $\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}}=\mathrm{Id}$ and
- if $\Lambda'=\{1,\dots,d\}$, $\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}}$ is just $\mathcal{J}_{z,\gamma}$.

Let $z'=-\mathbf{1}\in\mathbf{T}^{(d')}$ in $\mathcal{J}^{(\Lambda')}_{z',\gamma_{\Lambda'}}$ we obtain the partial Hankel-Clifford transform with respect to $x_{\Lambda'}=(x_{\lambda'_1},\dots,x_{\lambda'_{d'}})$ denoted by $\mathcal{H}^{(\Lambda')}_{\gamma_{\Lambda'}}$.

As a direct consequence of Corollary 3.2.1 we have the following result.

Corollary 3.2.2. *$\mathcal{J}^{(\Lambda')}_{z',0}$ is a topological isomorphism on $G^\alpha_\alpha(\mathbb{R}^d_+)$ with an inverse $\mathcal{J}^{(\Lambda')}_{\bar{z}',0}$. In particular, $\mathcal{H}^{(\Lambda')}_0$ is a self-inverse topological isomorphism on $G^\alpha_\alpha(\mathbb{R}^d_+)$.*

3.3 Fourier-Laguerre coefficients in $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$

In this section, we characterise the space $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ in terms of the Fourier-Laguerre coefficients.

Proposition 3.3.1. *([9, Lemma 3.1], for d=1) Let $a_n = \int_{\mathbb{R}_+^d} f(t)\mathcal{L}_n(t)dt$, $n \in \mathbb{N}_0^d$ and $f \in L^2(\mathbb{R}_+^d)$. If there exist constants $c > 0$ and $a > 1$ such that*

$$|a_n| \leq c a^{-|n|^{1/\alpha}},\ n \in \mathbb{N}_0^d, \tag{3.10}$$

then $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$.

Proof. As $\{a_n\}_{n\in\mathbb{N}_0^d} \in s^\alpha \subseteq s$, it follows $f \in \mathcal{S}(\mathbb{R}_+^d)$ and the series $\sum_n a_n \mathcal{L}_n$ converges absolutely in $\mathcal{S}(\mathbb{R}_+^d)$ to f. Since $n_1^{1/\alpha} + \ldots + n_d^{1/\alpha} \leq d|n|^{1/\alpha}$, denoting $\tilde{a} = a^{1/d} > 1$, we have

$$a^{-|n|^{1/\alpha}} \leq \prod_{l=1}^{d} \tilde{a}^{-n_l^{1/\alpha}}.$$

Using the estimates (1.6) and (1.7) for $p \in \mathbb{N}_0^d$, we have

$$\left\|t^{p/2}\mathcal{L}_n(t)\right\|_2 \leq 2^{|p|+5d}\prod_{l=1}^{d}(n_l+1)\ldots\left(n_l+\left[\frac{p_l}{2}\right]+2\right), \tag{3.11}$$

$$\left\|t^{p/2}D^p\mathcal{L}_n(t)\right\|_2 \leq 2^{5d}\prod_{l=1}^{d}(n_l+1)\ldots\left(n_l+\left[\frac{p_l}{2}\right]+2\right). \tag{3.12}$$

Let $\Lambda = \{\lambda_1, \ldots, \lambda_{d'}\} \subseteq \{1, \ldots, d\}$. Since

$$\mathcal{H}_0^{(\Lambda)}\mathcal{L}_n = (-1)^{n_{\lambda_1}+\ldots+n_{\lambda_{d'}}}\mathcal{L}_n,$$

(3.11) implies

$$\begin{aligned}
&\left\|t^{p/2}\mathcal{H}_0^{(\Lambda)}f(t)\right\|_2 \leq \sum_{n\in\mathbb{N}_0^d}|a_n|\left\|t^{p/2}\mathcal{L}_n(t)\right\|_2 \\
\leq\ & c2^{|p|+5d}\sum_{n\in\mathbb{N}_0^d}\prod_{l=1}^{d}\tilde{a}^{-n_l^{1/\alpha}}(n_l+1)\ldots\left(n_l+\left[\frac{p_l}{2}\right]+2\right) \\
\leq\ & c2^{|p|+5d}\prod_{l=1}^{d}\tilde{a}^{([p_l/2]+2)}\sum_{n\in\mathbb{N}_0^d}\prod_{l=1}^{d}\tilde{a}^{-(n_l+[p_l/2]+2)^{1/\alpha}}\left(n_l+\left[\frac{p_l}{2}\right]+2\right)^{[p_l/2]+2}.
\end{aligned}$$

Let $u > 0$, $v > 1$. Clearly, $\rho_{u,v}(x) = v^{-(x+u)^{1/\alpha}}(x+u)^u$, $x \in (-u, +\infty)$ attains its maximum at $x = (\alpha u/\ln v)^\alpha - u$. This implies that there exist $C_1, A_1 > 0$ such that

$$\left\|t^{p/2}\mathcal{H}_0^{(\Lambda)}f(t)\right\|_2 \leq C_1 A_1^{|p|}p^{(\alpha/2)p},\ \text{ for all }\ p \in \mathbb{N}_0^d,\ \Lambda \subseteq \{1, \ldots, d\}. \tag{3.13}$$

Similarly, by using (3.12), there exist $C_2, A_2 > 0$ such that

$$\left\| t^{p/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C_2 A_2^{|p|} p^{(\alpha/2)p}, \text{ for all } p \in \mathbb{N}_0^d,\ \Lambda \subseteq \{1, \ldots, d\}. \tag{3.14}$$

Since $\mathcal{H}_0^{(\Lambda)} f \in \mathcal{S}(\mathbb{R}_+^d)$, by integration by parts one easily verifies that

$$\begin{aligned} & \left| \langle t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t), t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \rangle_{L^2(\mathbb{R}_+^d)} \right| \\ = & \left| \langle D^p (t^{p+k} D^p \mathcal{H}_0^{(\Lambda)} f(t)), \mathcal{H}_0^{(\Lambda)} f(t) \rangle_{L^2(\mathbb{R}_+^d)} \right|. \end{aligned}$$

Hence, for all $k, p \in \mathbb{N}_0^d$ such that $2k \geq p$, by (3.13) and (3.14), we obtain

$$\begin{aligned} & \left\| t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2^2 \\ \leq & \sum_{m \leq p} \binom{p}{m} \frac{(p+k)!}{(p+k-m)!} \left| \left(t^{p+k-m} D^{2p-m} \mathcal{H}_0^{(\Lambda)} f(t), \mathcal{H}_0^{(\Lambda)} f(t) \right) \right| \\ \leq & \ 2^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m! \left| \left(t^{(2p-m)/2} D^{2p-m} \mathcal{H}_0^{(\Lambda)} f(t), t^{(2k-m)/2} \mathcal{H}_0^{(\Lambda)} f(t) \right) \right| \\ \leq & \ C' A'^{|p|+|k|} \sum_{m \leq p} \binom{p}{m} m^{(\alpha/2)m} m^{(\alpha/2)m} (2p-m)^{(\alpha/2)(2p-m)} (2k-m)^{(\alpha/2)(2k-m)} \\ \leq & \ C' A'^{|p|+|k|} 2^{|p|} (2p)^{\alpha p} (2k)^{\alpha k}, \end{aligned}$$

i.e. there exist $C_3, A_3 > 0$ such that for all $k, p \in \mathbb{N}_0^d$ such that $2k \geq p$ and all $\Lambda \subseteq \{1, \ldots, d\}$

$$\left\| t^{(p+k)/2} D^p \mathcal{H}_0^{(\Lambda)} f(t) \right\|_2 \leq C_3 A_3^{|p+k|} p^{(\alpha/2)p} k^{(\alpha/2)k}. \tag{3.15}$$

Let now $p, k \in \mathbb{N}_0^d$ be arbitrary but fixed. Let

$$\Lambda' = \{\lambda_1', \ldots, \lambda_{d'}'\} \subseteq \{1, \ldots, d\} \text{ be such that } k_{\lambda_l'} < \frac{p_{\lambda_l'}}{2},\ l = 1, \ldots, d'$$

and

$$\Lambda'' = \{\lambda_1'', \ldots, \lambda_{d''}''\} = \{1, \ldots, d\} \backslash \Lambda' \text{ be such that } k_{\lambda_l''} \geq \frac{p_{\lambda_l''}}{2},\ l = 1, \ldots, d''.$$

Then (3.9) and (3.15) imply

$$\begin{aligned} \left\| t^{(p+k)/2} D_t^p f(t) \right\|_2 & \leq 2^{|k|} \left\| t^{(p+k)/2} D_{t_{\Lambda'}}^{k_{\Lambda'}} D_{t_{\Lambda''}}^{p_{\Lambda''}} \mathcal{H}_0^{(\Lambda')} f(t) \right\|_2 \\ & \leq C_3 (2A_3)^{|p+k|} p^{(\alpha/2)p} k^{(\alpha/2)k}, \end{aligned}$$

i.e. $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$. □

Our next goal is to prove that $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ implies (3.10). We need some preparations.

Let $\mathbf{\Pi} = \Pi_1 \times ... \times \Pi_d$, where

$$\Pi_l = \{z_l \in \mathbb{C} |\, \operatorname{Im} z_l < 0\}, \qquad l = 1, ..., d.$$

One easily verifies that for each $z = x + iy \in \mathbf{\Pi}$, the functions

- $t \mapsto e^{-2\pi izt}$, $\mathbb{R}^d_+ \to \mathbb{C}$,
- $t \mapsto D_{x_l} e^{-2\pi i(x+iy)t} = -2\pi i t_l e^{-2\pi i(x+iy)t}$, $\mathbb{R}^d_+ \to \mathbb{C}$ for $l = 1, \ldots, d$ and
- $t \mapsto D_{y_l} e^{-2\pi i(x+iy)t} = 2\pi t_l e^{-2\pi i(x+iy)t}$, $\mathbb{R}^d_+ \to \mathbb{C}$ for $l = 1, \ldots, d$

are in $G^\alpha_\alpha(\mathbb{R}^d_+)$ (also in $\mathcal{S}(\mathbb{R}^d_+)$). For the moment, denote by e_l, $l = 1, \ldots, d$, the point in $\mathbb{R}^d$ such that all coordinates are 0 except the l-th coordinate which is equal to 1. By standard arguments, one proves that for the fixed $x^{(0)} = (x^{(0)}_1, \ldots, x^{(0)}_d) \in \mathbb{R}^d$ and $y^{(0)} = (y^{(0)}_1, \ldots, y^{(0)}_d) \in \mathbb{R}^d$ with $y^{(0)}_l < 0$, $l = 1, \ldots, d$ (i.e. $z^{(0)} = x^{(0)} + iy^{(0)} \in \mathbf{\Pi}$) we have

$$\left(e^{-2\pi i(x^{(0)}+x_l e_l+iy^{(0)})t} - e^{-2\pi i(x^{(0)}+iy^{(0)})t}\right)\frac{1}{x_l} \quad \to \quad -2\pi i t_l e^{-2\pi i(x^{(0)}+iy^{(0)})t},$$

as $x_l \to 0$ in $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ for some $A > 0$ and consequently in $G^\alpha_\alpha(\mathbb{R}^d_+)$ and $\mathcal{S}(\mathbb{R}^d_+)$. Also,

$$\left(e^{-2\pi i(x^{(0)}+i(y^{(0)}+y_l e_l))t} - e^{-2\pi i(x^{(0)}+iy^{(0)})t}\right)\frac{1}{y_l} \quad \to \quad 2\pi t_l e^{-2\pi i(x^{(0)}+iy^{(0)})t},$$

as $y_l \to 0$ in $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ for some $A > 0$ and consequently in $G^\alpha_\alpha(\mathbb{R}^d_+)$ and $\mathcal{S}(\mathbb{R}^d_+)$. Moreover,

$$-2\pi i t_l e^{-2\pi i(x+iy)t} \to -2\pi i t_l e^{-2\pi i(x^{(0)}+iy^{(0)})t}$$

and

$$2\pi t_l e^{-2\pi i(x+iy)t} \to 2\pi t_l e^{-2\pi i(x^{(0)}+iy^{(0)})t}$$

as $(x, y) \to (x^{(0)}, y^{(0)})$ in $G^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ for some $A > 0$. Hence, the same holds in $G^\alpha_\alpha(\mathbb{R}^d_+)$ and $\mathcal{S}(\mathbb{R}^d_+)$. It follows that for each $u \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$ or $u \in (\mathcal{S}(\mathbb{R}^d_+))'$, the function

$$z \mapsto \mathcal{F}_{\mathbf{\Pi}} u(z) = \langle u(t), e^{-2\pi izt}\rangle, \ \mathbf{\Pi} \to \mathbb{C},$$

is of the class $\mathcal{C}^1$;

$$D_{x_l}\mathcal{F}_{\mathbf{\Pi}} u(x + iy) = \langle u(t), D_{x_l} e^{-2\pi i(x+iy)t}\rangle$$

and

$$D_{y_l}\mathcal{F}_{\mathbf{\Pi}} u(x + iy) = \langle u(t), D_{y_l} e^{-2\pi i(x+iy)t}\rangle.$$

Since the Cauchy-Riemann equations hold for $\mathcal{F}_{\mathbf{\Pi}} u$, it is analytic on $\mathbf{\Pi}$.

Let $\mathbf{D} = D_1 \times ... \times D_d$, where

$$D_l = \{w_l \in \mathbb{C} |\, |w_l| < 1\}, \qquad l = 1, ..., d.$$

Observe that the mapping

$$w \mapsto \Omega(w) = \left(\frac{1 + w_1}{4\pi i(1 - w_1)}, \ldots, \frac{1 + w_d}{4\pi i(1 - w_d)}\right)$$

is a biholomorphic mapping from $\mathbf{D}$ onto $\mathbf{\Pi}$ with an inverse

$$z \mapsto \Omega^{-1}(z) = \left(\frac{4\pi i z_1 - 1}{4\pi i z_1 + 1}, \ldots, \frac{4\pi i z_d - 1}{4\pi i z_d + 1}\right).$$

Thus, we have the following result.

Lemma 3.3.1. *For each $u \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$ or $u \in (\mathcal{S}(\mathbb{R}_+^d))'$, the function*

$$\mathcal{F}_{\mathbf{D}}u(w) = \mathcal{F}_{\mathbf{\Pi}}u(\Omega(w)) = \left\langle u(t), \prod_{l=1}^{d} e^{-\frac{1}{2}\frac{1+w_l}{1-w_l}t_l} \right\rangle, \ \mathbf{D} \to \mathbb{C},$$

is analytic on $\mathbf{D}$, i.e. $\mathcal{F}_{\mathbf{D}}u \in \mathcal{O}(\mathbf{D})$.

Proposition 3.3.2. *([7, Proposition 1.1], for d=1) Let $u \in (\mathcal{S}(\mathbb{R}_+^d))'$ and $a_n = \langle u, \mathcal{L}_n \rangle$, $n \in \mathbb{N}_0^d$. Then,*

$$\mathcal{F}_{\mathbf{D}}(u)(w) = \prod_{j=1}^{d}(1-w_j) \sum_{n \in \mathbb{N}_0^d} a_n w^n, \ w \in \mathbf{D}. \tag{3.16}$$

In particular, if $\mathcal{F}_{\mathbf{D}}u = 0$ then $u = 0$.

Proof. By Theorem 2.3.1, $u = \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n$ and the series converges absolutely in $(\mathcal{S}(\mathbb{R}_+^d))'$. As $e^{-2\pi i z t} \in \mathcal{S}(\mathbb{R}_+^d)$, $z \in \mathbf{\Pi}$, we obtain

$$\mathcal{F}_{\mathbf{\Pi}}(u)(z) = \sum_{n \in \mathbb{N}_0^d} a_n \int_{\mathbb{R}_+^d} \mathcal{L}_n(t) e^{-2\pi i z t} dt, \ z \in \mathbf{\Pi}.$$

Using (1.13), we obtain

$$\mathcal{F}_{\mathbf{\Pi}}(u)(z) = \sum_{n \in \mathbb{N}_0^d} a_n \prod_{j=1}^{d} \frac{(\frac{1}{2} + 2\pi i z_j - 1)^{n_j}}{(\frac{1}{2} + 2\pi i z_j)^{n_j+1}}, \ z \in \mathbf{\Pi}.$$

By the definition of $\mathcal{F}_{\mathbf{D}}u$, (3.16) follows. $\square$

The next two assertions are already proved in [9], Lemma 3.2 and Corollary 3.5, in the case d=1. However, there are subtle gaps which we improve upon.

Proposition 3.3.3. *Let $\alpha \geq 1$ and $\{a_n\}_{n \in \mathbb{N}_0^d}$ be a sequence of complex numbers such that $a_n \to 0$ as $|n| \to \infty$. Then*

$$F(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_{n \in \mathbb{N}_0^d} a_n w^n, \ w \in \mathbf{D},$$

belongs to $\mathcal{O}(\mathbf{D})$. The following conditions are equivalent:

(*i*) *There exist constants $C, A > 0$ such that*

$$|D^p F(w)| \leq C A^{|p|} p^{\alpha p}, \quad p \in \mathbb{N}_0^d, w \in \mathbf{D}. \tag{3.17}$$

(*ii*) *There exist constants $c > 0$, $a > 1$ such that $|a_n| \leq c a^{-|n|^{1/\alpha}}$, $n \in \mathbb{N}_0^d$.*

Proof. Clearly $F \in \mathcal{O}(\mathbf{D})$. Let $\sum_{n\in\mathbb{N}_0^d} b_n w^n$ be the power series expansion of F at 0. Then, for $n \in \mathbb{N}_0^d$ we have

$$\begin{aligned} b_n &= \frac{D^n F(0)}{n!} = \frac{1}{n!} \sum_{\substack{k\leq n \\ k\leq \mathbf{1}}} \binom{n}{k} (-1)^{|k|} \left(\sum_{m\geq n-k} \frac{m!}{(m-n+k)!} a_m w^{m-n+k} \right) \Bigg|_{w=0} \\ &= \sum_{\substack{k\leq n \\ k\leq \mathbf{1}}} (-1)^{|k|} a_{n-k}. \end{aligned} \tag{3.18}$$

Thus, for $n, m \in \mathbb{N}_0^d$,

$$\sum_{p\leq m} b_{n+\mathbf{1}+p} = \sum_{p\leq m} \sum_{k\leq \mathbf{1}} (-1)^{|k|} a_{n+p+\mathbf{1}-k}. \tag{3.19}$$

Firstly, assume that $d \geq 2$. Denote by $Q_{n,m}$ the d-dimensional parallelepiped

$$Q_{n,m} = \{x \in \mathbb{R}^d |\, n_l \leq x_l \leq n_l + m_l + 1,\ l = 1, \ldots, d\}.$$

If $q \in \mathbb{N}_0^d$ is such that:

- $n + q$ is in the interior of $Q_{n,m}$
  Then a_{n+q} appears exactly 2^d times in the sum on the right hand side of (3.19) such that 2^{d-1} times with the "+" sign and 2^{d-1} times with "−" sign.

- $n + q$ is on the s-dimensional face of $Q_{n,m}$, $1 \leq s \leq d - 1$
  Then a_{n+q} appears exactly 2^s times half of which are with the "+" sign and the other half with the "−" sign.

Thus on the right hand side of (3.19) everything cancels except for those terms which indexes are the vertices of $Q_{n,m}$ and they appear only once. For $k \in \mathbb{N}_0^d$ with $k \leq \mathbf{1}$ denote by $m^{(k)}$ the multi-index that satisfies

$$m_l^{(k)} = \begin{cases} 0, & k_l = 0 \\ m_l + 1, & k_l = 1 \end{cases}$$

$l = 1, \ldots, d$; when k varies through the multi-indexes that are $\leq \mathbf{1}$, $n + m + \mathbf{1} - m^{(k)}$ varies through the vertices of $Q_{n,m}$. Using this notations, by the above observations, we have

$$\sum_{p\leq m} b_{n+\mathbf{1}+p} = \sum_{k\leq \mathbf{1}} (-1)^{|k|} a_{n+m+\mathbf{1}-m^{(k)}},\ \forall n, m \in \mathbb{N}_0^d. \tag{3.20}$$

Clearly, for $d = 1$ (3.19) and (3.20) are equal.

Assume that (i) holds. Since

$$D^p F(w) = \sum_{n\geq p} \frac{n!}{(n-p)!} b_n w^{n-p},$$

the hypothesis in (i) and the Cauchy formula yield

$$\frac{n!}{(n-p)!}|b_n| \leq CA^{|p|}p^{\alpha p}, \text{ for all } n,p \in \mathbb{N}_0^d,\ n \geq p.$$

As $n!/(n-p)! \geq e^{-|p|}n^p$, for $n \geq p$, we have

$$|b_n| \leq C \prod_{j=1}^{d} \inf_{p_j \leq n_j} \frac{(eA)^{p_j} p_j^{\alpha p_j}}{n_j^{p_j}},\ n \in \mathbb{N}_0^d. \tag{3.21}$$

Of course we can assume $A \geq 1$. Then, if $p_j \geq n_j$,

$$\frac{(eA)^{p_j} p_j^{\alpha p_j}}{n_j^{p_j}} \geq \frac{(eA)^{n_j} n_j^{\alpha n_j}}{n_j^{n_j}},$$

and so the infimum in (3.21) can be taken varying on $p_j \geq 0$, $j = 1, \ldots, d$. Thus, [12, (2) and (3), p. 169-170] imply, with suitable $c' > 0$ and $a' > 1$,

$$|b_n| \leq C \prod_{j=1}^{d} \inf_{p_j \geq 0} \frac{(eA)^{p_j} p_j^{\alpha p_j}}{n_j^{p_j}} \leq c' a'^{-|n|^{1/\alpha}},\ n \in \mathbb{N}_0^d.$$

Observe that for $p, n \in \mathbb{N}_0^d$ with $p \geq n$, we have

$$|p|^{1/\alpha} \geq \frac{1}{2}(|p-n|^{1/\alpha} + |n|^{1/\alpha}).$$

Thus, if we put $a = \sqrt{a'} > 1$ we have

$$a'^{-|p|^{1/\alpha}} \leq a^{-|p-n|^{1/\alpha}} a^{-|n|^{1/\alpha}}, \text{ for all } p \geq n.$$

The above estimate for $|b_n|$ together with (3.20) implies that for all $n, m \in \mathbb{N}_0^d$

$$\begin{aligned}
|a_n| &\leq \sum_{p \leq m} |b_{n+\mathbf{1}+p}| + \sum_{\substack{k \leq \mathbf{1} \\ k \neq \mathbf{1}}} |a_{n+m+\mathbf{1}-m^{(k)}}| \\
&\leq c' a^{-|n|^{1/\alpha}} \sum_{p \in \mathbb{N}_0^d} a^{-|p|^{1/\alpha}} + \sum_{\substack{k \leq \mathbf{1} \\ k \neq \mathbf{1}}} |a_{n+m+\mathbf{1}-m^{(k)}}| \\
&= c a^{-|n|^{1/\alpha}} + \sum_{\substack{k \leq \mathbf{1} \\ k \neq \mathbf{1}}} |a_{n+m+\mathbf{1}-m^{(k)}}|.
\end{aligned}$$

The last sum has exactly $2^d - 1$ terms and since $k \neq \mathbf{1}$,

$$|n + m + \mathbf{1} - m^{(k)}| \geq |n| + \min\{m_l |\, l = 1, \ldots, d\}.$$

Let $n \in \mathbb{N}_0^d$ be arbitrary but fixed. Since the above estimate for $|a_n|$ holds for arbitrary $m \in \mathbb{N}_0^d$ and since $a_n \to 0$ as $|n| \to \infty$ (by hypothesis), this implies $|a_n| \leq c a^{-|n|^{1/\alpha}}$.

Assume now that (ii) holds. Then (3.18) implies the existence of $a > 1$ and $c > 0$ such that $|b_n| \leq ca^{-|n|^{1/\alpha}}$, $\forall n \in \mathbb{N}_0^d$. Observe that $n_1^{1/\alpha} + \ldots + n_d^{1/\alpha} \leq d|n|^{1/\alpha}$. Hence, by putting $a' = a^{1/d}$, we have

$$a^{-|n|^{1/\alpha}} \leq \prod_{j=1}^{d} a'^{-n_j^{1/\alpha}}.$$

Now, for $p \in \mathbb{N}_0^d$ and $w \in \mathbf{D}$ we obtain

$$|D^p F(w)| \leq \sum_{n \geq p} \frac{n!}{(n-p)!} |b_n| \leq c \sum_{n \in \mathbb{N}_0^d} \prod_{j=1}^{d} n_j^{p_j} a'^{-n_j^{1/\alpha}}.$$

Since $\rho(x) = x^p u^{-x^{1/\alpha}}$, $x \geq 0$ $(u > 1,\ p \in \mathbb{N}_0)$ attains its maximum at $x = (\alpha p / \ln u)^{\alpha}$, we proved (3.17). $\square$

We will prove in Proposition 3.3.5 that for $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$, the analytic function $\mathcal{F}_{\mathbf{D}}(f)$ satisfies part (i) of the previous proposition. In order to prove this we need the next result; its proof is analogous to the proof of [9, Theorem 3.3] for the one dimensional case and we omit it.

Proposition 3.3.4. *Let $f \in G_\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$. Then there exist constants $C, A > 0$ such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq CA^{|p|} p^{\alpha p},\ p \in \mathbb{N}_0^d,\ w \in \mathbf{D},\ \operatorname{Re} w_l \leq 0,\ l = 1, ..., d.$$

Proposition 3.3.5. *Let $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$. Then there exist constants $C, A > 0$ such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq CA^{|p|} p^{\alpha p},\ p \in \mathbb{N}_0^d,\ w \in \mathbf{D} \tag{3.22}$$

and $\lim_{w \to \mathbf{1}} \mathcal{F}_{\mathbf{D}}(f)(w) = 0$.

Proof. As $f \in \mathcal{S}(\mathbb{R}_+^d)$, Proposition 3.3.2 implies that $\lim_{w \to \mathbf{1}} \mathcal{F}_{\mathbf{D}}(f)(w) = 0$.

We introduce some notation to make the proof simpler. Let

$$\Lambda' = \{\lambda_1', \ldots, \lambda_{d'}'\} \subseteq \{1, \ldots, d\} \text{ and } \Lambda'' = \{\lambda_1'', \ldots, \lambda_{d''}''\} = \{1, \ldots, d\} \backslash \Lambda'.$$

For $\zeta \in \mathbb{C}^d$ (or in $\mathbb{R}_+^d$, or in $\mathbb{N}_0^d$), by abusing the notation, we write $\zeta = (\zeta_{\Lambda'}, \zeta_{\Lambda''})$ where

$$\zeta_{\Lambda'} = (\zeta_{\lambda_1'}, \ldots, \zeta_{\lambda_{d'}'}) \quad \text{and} \quad \zeta_{\Lambda''} = (\zeta_{\lambda_1''}, \ldots, \zeta_{\lambda_{d''}''}).$$

Let $\tilde{\Lambda}'$ be the biholomorphic mapping from $\mathbb{C}^d$ onto itself defined by $\tilde{\Lambda}' w = \zeta$ where

$$\zeta_{\lambda_l'} = -w_{\lambda_l'},\ l = 1, \ldots, d' \text{ and } \zeta_{\lambda_s''} = w_{\lambda_s''},\ s = 1, \ldots, d''.$$

Also, denote

$$\mathbf{D}_{(\Lambda')} = \{\zeta \in \mathbf{D} |\, \operatorname{Re} \zeta_{\lambda_l'} \geq 0,\ l = 1, \ldots, d',\ \text{and } \operatorname{Re} \zeta_{\lambda_s''} \leq 0,\ s = 1, \ldots, d''\}$$

(note that $\mathbf{D}_{(\emptyset)}$ consists of all $w \in \mathbf{D}$ such that the coordinates of w have non-positive real parts).

For $f \in \mathcal{S}(\mathbb{R}_+^d)$ let $a_n = \langle f, \mathcal{L}_n\rangle$, $n \in \mathbb{N}_0^d$. Then, Proposition 3.3.2 implies $\mathcal{F}_{\mathbf{D}} f(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_{n\in\mathbb{N}_0^d} a_n w^n$, $w \in \mathbf{D}$. As

$$\langle \mathcal{H}_0^{(\Lambda')} f, \mathcal{L}_n\rangle = (-1)^{n_{\lambda'_1}+\ldots+n_{\lambda'_{d'}}} a_n$$

we obtain

$$\mathcal{F}_{\mathbf{D}}(\mathcal{H}_0^{(\Lambda')} f)(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_{n\in\mathbb{N}_0^d} (-1)^{n_{\lambda'_1}+\ldots+n_{\lambda'_{d'}}} a_n w^n, \;\; w \in \mathbf{D}.$$

Now,

$$\mathcal{F}_{\mathbf{D}}(\mathcal{H}_0^{(\Lambda')} f)(\tilde{\Lambda}' w) = \left(\prod_{l=1}^{d'} \frac{1 + w_{\lambda'_l}}{1 - w_{\lambda'_l}}\right) \cdot (\mathbf{1} - w)^{\mathbf{1}} \sum_{n\in\mathbb{N}_0^d} a_n w_n, \;\; w \in \mathbf{D}.$$

Thus

$$\mathcal{F}_{\mathbf{D}} f(w) = \left(\prod_{l=1}^{d'} \frac{1 - w_{\lambda'_l}}{1 + w_{\lambda'_l}}\right) \mathcal{F}_{\mathbf{D}}(\mathcal{H}_0^{(\Lambda')} f)(\tilde{\Lambda}' w), \;\; w \in \mathbf{D},\; f \in \mathcal{S}(\mathbb{R}_+^d). \tag{3.23}$$

Let $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$. Since $\mathcal{H}_0^{(\Lambda')} f \in G_\alpha^\alpha(\mathbb{R}_+^d)$ (cf. Corollary 3.2.2), Proposition 3.3.4 implies the existence of $A, C > 0$ such that

$$\left| D^n \mathcal{F}_{\mathbf{D}}(\mathcal{H}_0^{(\Lambda')} f)(w)\right| \leq C A^{|n|} n^{\alpha n}, \;\; \forall n \in \mathbb{N}_0^d, \forall w \in \mathbf{D}_{(\emptyset)}, \forall \Lambda' \subseteq \{1, \ldots, d\}. \tag{3.24}$$

Observe that for $w \in \mathbf{D}_{(\emptyset)}$, (3.22) holds by Proposition 3.3.4. To prove (3.22) for $w \in \mathbf{D}_{(\Lambda')}$ when $\emptyset \neq \Lambda' \subseteq \{1, \ldots, d\}$, we need an estimate for the derivatives of the function

$$\zeta \mapsto \frac{1-\zeta}{1+\zeta}, \; \{\zeta \in \mathbb{C} |\, |\zeta| < 1\} \to \mathbb{C} \text{ when } \operatorname{Re}\zeta \geq 0.$$

Since $(1-\zeta)/(1+\zeta) = 2/(1+\zeta) - 1$ and $|1+\zeta| \geq 1$ when $\operatorname{Re}\zeta \geq 0$, for $j \in \mathbb{N}$ we have

$$\left| \frac{d^j}{d\zeta^j}\left(\frac{1-\zeta}{1+\zeta}\right)\right| = \frac{2j!}{|1+\zeta|^{j+1}} \leq 2j!, \;\; \text{when } \; |\zeta| < 1 \text{ and } \operatorname{Re}\zeta \geq 0. \tag{3.25}$$

Clearly, (3.25) also holds for $j = 0$. Now, observe that $\tilde{\Lambda}'(\mathbf{D}_{(\Lambda')}) = \mathbf{D}_{(\emptyset)}$. Hence, for $w \in \mathbf{D}_{(\Lambda')}$, (3.23), (3.24) and (3.25) imply

$$\begin{aligned}
|D^n \mathcal{F}_{\mathbf{D}} f(w)| &\leq \sum_{m_{\Lambda'}\leq n_{\Lambda'}} \binom{n_{\Lambda'}}{m_{\Lambda'}} 2^{d'} m_{\Lambda'}! \left| D_{w_{\Lambda'}}^{n_{\Lambda'}-m_{\Lambda'}} D_{w_{\Lambda''}}^{n_{\Lambda''}} \mathcal{F}_{\mathbf{D}}(\mathcal{H}_0^{(\Lambda')} f)(\tilde{\Lambda}' w)\right| \\
&\leq C_1 \sum_{m_{\Lambda'}\leq n_{\Lambda'}} \binom{n_{\Lambda'}}{m_{\Lambda'}} m_{\Lambda'}^{\alpha m_{\Lambda'}} A^{|n|-|m_{\Lambda'}|} (n_{\Lambda'} - m_{\Lambda'})^{\alpha(n_{\Lambda'}-m_{\Lambda'})} n_{\Lambda''}^{\alpha n_{\Lambda''}} \\
&\leq C_1 (2A)^{|n|} n^{\alpha n},
\end{aligned}$$

which completes the proof. □

Now, Proposition 3.3.1, Proposition 3.3.5, Proposition 3.3.2 and Proposition 3.3.3 give the main result of this section:

Theorem 3.3.1. *([9, Theorem 3.6], for d=1) Let $\alpha \geq 1$. For $f \in L^2(\mathbb{R}^d_+)$ let*

$$a_n = \int_{\mathbb{R}^d_+} f(t)\mathcal{L}_n(t)dt, \quad n \in \mathbb{N}_0^d.$$

The following conditions are equivalent:

(i) *There exist $c > 0$ and $a > 1$ such that*

$$|a_n| \leq ca^{-|n|^{1/\alpha}} \quad \text{for } n \in \mathbb{N}_0^d.$$

(ii) *$f \in G^\alpha_\alpha(\mathbb{R}^d_+)$.*

(iii) *There exist $C, A > 0$ such that*

$$|D^p \mathcal{F}_{\mathbf{D}}(f)(w)| \leq CA^{|p|}p^{\alpha p} \quad \text{for } p \in \mathbb{N}_0^d \quad \text{and } w \in \mathbf{D}$$

and $\lim_{w \to \mathbf{1}} \mathcal{F}_{\mathbf{D}}(f)(w) = 0$.

Conversely, given a sequence $\{a_n\}_{n \in \mathbb{N}_0^d}$ satisfying condition (i) or given $F \in \mathcal{O}(\mathbf{D})$ of the form $F(w) = (\mathbf{1} - w)^{\mathbf{1}} \sum_n a_n w^n$ with $a_n \to 0$ as $|n| \to \infty$ which satisfies (3.17), there exists $f \in G^\alpha_\alpha(\mathbb{R}^d_+)$ such that $a_n = \int_{\mathbb{R}^d_+} f(t)\mathcal{L}_n(t)dt$ and $\mathcal{F}_{\mathbf{D}}(f)(w) = F(w)$ for $w \in \mathbf{D}$.

3.4 Topological properties of $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$

As we shell see, we gain deep insights into the topological structure of $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ by Theorem 3.3.1. Let $\iota : G^\alpha_\alpha(\mathbb{R}^d_+) \to s^\alpha$, $\iota(f) = \{\langle f, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$. Theorem 3.3.1 proves that ι is a well defined bijection.

Theorem 3.4.1. *Let $\alpha \geq 1$. The mapping*

$$\iota : G^\alpha_\alpha(\mathbb{R}^d_+) \to s^\alpha, \ \iota(f) = \{\langle f, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$$

is a topological isomorphism between $G^\alpha_\alpha(\mathbb{R}^d_+)$ and s^α.

In particular, $G^\alpha_\alpha(\mathbb{R}^d_+)$ is a (DFN)-space and $(G^\alpha_\alpha(\mathbb{R}^d_+))'$ is an (FN)-space.

For each $f \in G^\alpha_\alpha(\mathbb{R}^d_+)$, $\sum_{n \in \mathbb{N}_0^d} \langle f, \mathcal{L}_n \rangle \mathcal{L}_n$ is summable to f in $G^\alpha_\alpha(\mathbb{R}^d_+)$.

Proof. If we consider ι as a linear mapping from $G^\alpha_\alpha(\mathbb{R}^d_+)$ into s (s^α is canonically injected into s) then ι is continuous since it decomposes as

$$G^\alpha_\alpha(\mathbb{R}^d_+) \longrightarrow \mathcal{S}(\mathbb{R}^d_+) \xrightarrow{f \mapsto \{\langle f, \mathcal{L}_n \rangle\}_n} s,$$

where the first mapping is the canonical inclusion. Hence, ι has a closed graph in $G^\alpha_\alpha(\mathbb{R}^d_+) \times s$. Since the range of ι is in s^α and s^α is continuously injected into s, the graph of ι is closed in $G_\alpha^\alpha(\mathbb{R}_+^d) \times s^\alpha$. $G_\alpha^\alpha(\mathbb{R}_+^d)$ is an injective inductive limit of (F)-spaces. For this reason, $G_\alpha^\alpha(\mathbb{R}_+^d)$ is ultrabornological (see Proposition A.8.3; every (F)-space is ultrabornological). Moreover, s^α is a webbed space of De Wilde (see Proposition A.8.2). Hence, the closed graph theorem of De Wilde (see Theorem A.8.1) implies that $\iota : G_\alpha^\alpha(\mathbb{R}_+^d) \to s^\alpha$ is continuous.

Also, s^α is ultrabornological since it is bornological and complete and $G_\alpha^\alpha(\mathbb{R}_+^d)$ is a webbed space of De Wilde (see Proposition A.8.1; every (F)-space is a webbed space of De Wilde). The mapping $\iota^{-1} : s^\alpha \to G_\alpha^\alpha(\mathbb{R}_+^d)$, which has a closed graph, is continuous by the De Wilde closed graph theorem (see Theorem A.8.1).

Now, Proposition 1.3.1 implies that $G_\alpha^\alpha(\mathbb{R}_+^d)$ is a (DFN)-space and $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ is an (FN)-space.

Given $f \in G_\alpha^\alpha(\mathbb{R}_+^d)$, let $a_n = \langle f, \mathcal{L}_n \rangle$. For each finite $\Phi \subseteq \mathbb{N}_0^d$, denote

$$f_\Phi = \sum_{n \in \Phi} a_n \mathcal{L}_n \in G_\alpha^\alpha(\mathbb{R}_+^d)$$

(since $\mathcal{L}_n \in G_\alpha^\alpha(\mathbb{R}_+^d)$). Let $a > 1$ be such that $\iota(f) \in s^{\alpha,a}$. Fix $1 < a' < a$. One easily verifies that for each $\varepsilon > 0$ there exists finite $\Phi_0 \subseteq \mathbb{N}_0^d$ such that for each finite $\Phi \subseteq \mathbb{N}_0^d$, satisfying $\Phi_0 \subseteq \Phi$, we have

$$\|\iota(f) - \iota(f_\Phi)\|_{s^{\alpha,a'}} \le \varepsilon.$$

Since ι is an isomorphism this implies that for each neighbourhood of zero $V \subseteq G_\alpha^\alpha(\mathbb{R}_+^d)$ there exists finite $\Phi_0 \subseteq \mathbb{N}_0^d$ such that for finite $\Phi \supseteq \Phi_0$ we have $f - f_\Phi \in V$, i.e. $\sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n$ is summable to f in $G_\alpha^\alpha(\mathbb{R}_+^d)$. □

Theorem 3.4.2. *Let $\alpha \ge 1$. The mapping*

$$\tilde{\iota} : (G_\alpha^\alpha(\mathbb{R}_+^d))' \to (s^\alpha)',\ \tilde{\iota}(T) = \{\langle T, \mathcal{L}_n \rangle\}_{n \in \mathbb{N}_0^d}$$

is a topological isomorphism.

Moreover, $\sum_{n \in \mathbb{N}_0^d} \langle T, \mathcal{L}_n \rangle \mathcal{L}_n$ is summable to T in $(G_\alpha^\alpha(\mathbb{R}_+^d))'$.

Proof. By Theorem 3.4.1, both the transpose of ι, ${}^t\iota : (s^\alpha)' \to (G_\alpha^\alpha(\mathbb{R}_+^d))'$, and its inverse $({}^t\iota)^{-1} : (G_\alpha^\alpha(\mathbb{R}_+^d))' \to (s^\alpha)'$ are topological isomorphisms. For $T \in (G_\alpha^\alpha(\mathbb{R}_+^d))'$, let $\{b_n\}_n = ({}^t\iota)^{-1}(T)$. Then

$$\langle T, \mathcal{L}_n \rangle = \langle {}^t\iota(\{b_n\}_n), \mathcal{L}_n \rangle = \langle \{b_n\}_n, \iota(\mathcal{L}_n) \rangle = b_n.$$

Thus,

$$\{\langle T, \mathcal{L}_n \rangle\}_n = \{b_n\}_n = ({}^t\iota)^{-1}(T) \in (s^\alpha)'.$$

Hence, $\tilde{\iota}$ is in fact a topological isomorphism

$$({}^t\iota)^{-1} : (G_\alpha^\alpha(\mathbb{R}_+^d))' \to (s^\alpha)'.$$

By the similar approach as above, one proves that $\sum_{n \in \mathbb{N}_0^d} \langle T, \mathcal{L}_n \rangle \mathcal{L}_n$ is summable to T in $(G_\alpha^\alpha(\mathbb{R}_+^d))'$. □

For $T \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$, by Lemma 3.3.1, $\mathcal{F}_{\mathbf{D}}T \in \mathcal{O}(\mathbf{D})$. Since $\sum_n \langle T, \mathcal{L}_n\rangle \mathcal{L}_n$ is summable to T in $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, by the same method as in the proof of Proposition 3.3.2, one proves the following result.

Proposition 3.4.1. *Let $T \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$, $\alpha \geq 1$ and $b_n = \langle T, \mathcal{L}_n\rangle$, $n \in \mathbb{N}^d_0$. Then,*

$$\mathcal{F}_{\mathbf{D}}(T)(w) = \prod_{j=1}^{d}(1-w_j)\sum_{n\in\mathbb{N}^d_0} b_n w^n,\ w \in \mathbf{D}.$$

In particular, if $\mathcal{F}_{\mathbf{D}}T = 0$ then $T = 0$.

3.5 Kernel theorem for $G^\alpha_\alpha(\mathbb{R}^d_+)$ and its dual space

In this section we state the Schwartz kernel theorem for $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and its dual space. We review the topological tensor product theory in Appendix A.5.

Theorem 3.5.1. *Let $\alpha \geq 1$. We have the following canonical isomorphism:*

$$G^\alpha_\alpha(\mathbb{R}^{d_1}_+)\hat{\otimes}G^\alpha_\alpha(\mathbb{R}^{d_2}_+) \cong G^\alpha_\alpha(\mathbb{R}^{d_1+d_2}_+)$$

and

$$(G^\alpha_\alpha(\mathbb{R}^{d_1}_+))'\hat{\otimes}(G^\alpha_\alpha(\mathbb{R}^{d_2}_+))' \cong (G^\alpha_\alpha(\mathbb{R}^{d_1+d_2}_+))' \cong \mathcal{L}(G^\alpha_\alpha(\mathbb{R}^{d_2}_+), (G^\alpha_\alpha(\mathbb{R}^{d_1}_+))'). \tag{3.26}$$

Proof. For simplicity, put $d = d_1 + d_2$. Let $s^\alpha_{d_1}$, $s^\alpha_{d_2}$ and s^α be the d_1-dimensional, the d_2-dimensional and the d-dimensional variant of the space s^α, respectively.

Firstly, we prove that $(s^\alpha_{d_1})'\hat{\otimes}(s^\alpha_{d_2})' \cong (s^\alpha)'$, where an isomorphism is given by the extension of the canonical inclusion

$$(s^\alpha_{d_1})' \otimes (s^\alpha_{d_2})' \to (s^\alpha)',\ \ \{u_n\}_{n\in\mathbb{N}^{d_1}_0} \otimes \{v_m\}_{m\in\mathbb{N}^{d_2}_0} \mapsto \{u_n v_m\}_{(n,m)\in\mathbb{N}^d_0}.$$

Observe that the mapping

$$\Big(\{u_n\}_{n\in\mathbb{N}^{d_1}_0}, \{v_m\}_{m\in\mathbb{N}^{d_2}_0}\Big) \mapsto \{u_n v_m\}_{(n,m)\in\mathbb{N}^d_0},\ \ (s^\alpha_{d_1})' \times (s^\alpha_{d_2})' \to (s^\alpha)'$$

is continuous. Hence, the π topology on $(s^\alpha_{d_1})' \otimes (s^\alpha_{d_2})'$ is stronger than the induced one from $(s^\alpha)'$.

Let A and B be the equicontinuous subsets of $((s^\alpha_{d_1})')' = s^\alpha_{d_1}$ and $((s^\alpha_{d_2})')' = s^\alpha_{d_2}$, respectively ($s^\alpha$ is reflexive since it is a (DFN)-space). Hence, there exist $C > 0$ and $r > 1$ such that

$$|\langle \{u_n\}_{n\in\mathbb{N}^{d_1}_0}, \{a_n\}_{n\in\mathbb{N}^{d_1}_0}\rangle| \leq C \sum_{n\in\mathbb{N}^{d_1}_0} |u_n| r^{-|n|^{1/\alpha}}$$

for all $\{a_n\}_{n\in\mathbb{N}^{d_1}_0} \in A$ and for all $\{u_n\}_{n\in\mathbb{N}^{d_1}_0} \in (s^\alpha_{d_1})$ and

$$|\langle \{v_m\}_{m\in\mathbb{N}^{d_2}_0}, \{b_m\}_{m\in\mathbb{N}^{d_2}_0}\rangle| \leq C \sum_{m\in\mathbb{N}^{d_2}_0} |v_m| r^{-|m|^{1/\alpha}}$$

for $\{b_m\}_{m\in\mathbb{N}_0^{d_2}} \in B$ and $\{v_m\}_{m\in\mathbb{N}_0^{d_2}} \in (s^\alpha_{d_2})'$. Let

$$\{\chi_{(n,m)}\}_{(n,m)\in\mathbb{N}_0^d} = \sum_{j=1}^{l} \{u_n^{(j)}\}_{n\in\mathbb{N}_0^{d_1}} \otimes \{v_m^{(j)}\}_{m\in\mathbb{N}_0^{d_2}} \in (s^\alpha_{d_1})' \otimes (s^\alpha_{d_2})'.$$

Then, for $\{a_n\}_{n\in\mathbb{N}_0^{d_1}} \in A$ and $\{b_m\}_{m\in\mathbb{N}_0^{d_2}} \in B$, we have

$$\begin{aligned}
\left|\left\langle \{\chi_{(n,m)}\}_{(n,m)}, \{a_n\}_n \otimes \{b_m\}_m \right\rangle\right| &= \left|\langle \sum_{j=1}^{l} \{u_n^{(j)}\}_n \left\langle \{v_m^{(j)}\}_m, \{b_m\}_m \right\rangle, \{a_n\}_n \rangle\right| \\
= & \left|\left\langle \left\{ \sum_{j=1}^{l} \left\langle \{v_m^{(j)}\}_m, \{b_m\}_m \right\rangle u_n^{(j)} \right\}_n, \{a_n\}_n \right\rangle\right| \\
\leq & \; C \sum_{n\in\mathbb{N}_0^{d_1}} \left| \sum_{j=1}^{l} \left\langle \{v_m^{(j)}\}_m, \{b_m\}_m \right\rangle u_n^{(j)} \right| r^{-|n|^{1/\alpha}} \\
= & \; C \sum_{n\in\mathbb{N}_0^{d_1}} \left| \left\langle \left\{ \sum_{j=1}^{l} u_n^{(j)} v_m^{(j)} \right\}_m, \{b_m\}_m \right\rangle \right| r^{-|n|^{1/\alpha}} \\
\leq & \; C^2 \sum_{(n,m)\in\mathbb{N}^d} \left| \sum_{j=1}^{l} u_n^{(j)} v_m^{(j)} \right| r^{-|n|^{1/\alpha}-|m|^{1/\alpha}} \leq C^2 \|\{\chi_{(n,m)}\}_{(n,m)}\|_{(s^\alpha)',r}.
\end{aligned}$$

We can conclude that the ϵ topology on $(s^\alpha_{d_1})'\otimes(s^\alpha_{d_2})'$ is weaker than the induced one from $(s^\alpha)'$. Since $(s^\alpha)'$ is nuclear, these topologies are identical. Clearly, $(s^\alpha_{d^1})' \otimes (s^\alpha_{d_2})'$ is dense in $(s^\alpha_d)'$. Hence, we proved the desired topological isomorphism.

As all spaces in consideration are (FN)-spaces, by duality we have $s^\alpha_{d_1}\hat{\otimes} s^\alpha_{d_2} \cong s^\alpha$. Note that the isomorphism is in fact the extension of the canonical inclusion

$$\kappa : s^\alpha_{d_1} \otimes s^\alpha_{d_2} \to s^\alpha, \;\; \kappa(\{a_n\}_n \otimes \{b_m\}_m) = \{a_n b_m\}_{(n,m)}.$$

Now observe that the diagram

$$\begin{tikzcd}
s^\alpha_{d_1} \otimes s^\alpha_{d_2} \arrow[r, "\kappa"] & s^\alpha \\
G^\alpha_\alpha(\mathbb{R}^{d_1}_+) \otimes G^\alpha_\alpha(\mathbb{R}^{d_2}_+) \arrow[u, "\iota\otimes\iota"] \arrow[r] & G^\alpha_\alpha(\mathbb{R}^d_+) \arrow[u, "\iota"']
\end{tikzcd}$$

commutes, where the bottom horizontal line is the canonical inclusion $f\otimes g(x,y) \mapsto f(x)g(y)$. Since κ extends to an isomorphism, by Theorem 3.4.1, it follows that the canonical inclusion $G^\alpha_\alpha(\mathbb{R}^{d_1}_+)\otimes G^\alpha_\alpha(\mathbb{R}^{d_2}_+) \to G^\alpha_\alpha(\mathbb{R}^d_+)$ is continuous and it extends to an isomorphism $G^\alpha_\alpha(\mathbb{R}^{d_1}_+)\hat{\otimes} G^\alpha_\alpha(\mathbb{R}^{d_2}_+) \cong G^\alpha_\alpha(\mathbb{R}^d_+)$. The assertion

$$(G^\alpha_\alpha(\mathbb{R}^{d_1}_+))'\hat{\otimes}(G^\alpha_\alpha(\mathbb{R}^{d_2}_+))' \cong (G^\alpha_\alpha(\mathbb{R}^d_+))'$$

can be obtained by the duality of an isomorphism $G^\alpha_\alpha(\mathbb{R}^{d_1}_+)\hat{\otimes} G^\alpha_\alpha(\mathbb{R}^{d_2}_+) \cong G^\alpha_\alpha(\mathbb{R}^d_+)$ since $G^\alpha_\alpha(\mathbb{R}^d_+)$ is a (DFN)-space. $\square$

The isomorphism (3.26) calls for some comment:

To every $K(x,y) \in (G_\alpha^\alpha(\mathbb{R}_+^{d_1+d_2}))'$ we may associate a continuous linear mapping K of $G_\alpha^\alpha(\mathbb{R}_+^{d_2})$ into $(G_\alpha^\alpha(\mathbb{R}_+^{d_1}))'$ in the following manner: if $v \in G_\alpha^\alpha(\mathbb{R}_+^{d_2})$, then

$$(Kv)(x) = \int_{\mathbb{R}_+^{d_2}} K(x,y)v(y)dy \in (G_\alpha^\alpha(\mathbb{R}_+^{d_1}))'.$$

Theorem 3.5.1 states that the correspondence $K(x,y) \leftrightarrow K$ is an isomorphism.

3.6 Topological isomorphism between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$

We will be particularly interested in the subspace $\mathcal{S}_{\alpha,\,\mathrm{even}}^\alpha(\mathbb{R}^d)$ of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ consisting of all "even" functions in $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, i.e. of all $\psi \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ such that

$$\psi(x_1,\ldots,x_{j-1},-x_j,x_{j+1},\ldots,x_d) = \psi(x), \tag{3.27}$$

for all $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, $j = 1,\ldots,d$.

Proposition 3.6.1. *The space $\mathcal{S}_{\alpha,\,\mathrm{even}}^\alpha(\mathbb{R}^d)$ is a closed subspace of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$. In particular, it is a (DFS)-space. Moreover, $\mathcal{S}_{\alpha,\,\mathrm{even}}^\alpha(\mathbb{R}^d)$ consists of those $\psi \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ which can be represented as $\psi = \sum_{n\in\mathbb{N}_0^d} a_{2n}h_{2n}$ where $\{a_{2n}\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$.*

Remark 3.6.1. Before we give the proof of this proposition, we want to explain the meaning of $\{a_{2n}\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$. It should be understood as the sequence $\{b_k\}_{k\in\mathbb{N}_0^d} \in s^{2\alpha}$ such that the elements with indexes $k = 2n$, $n \in \mathbb{N}_0^d$, are equal to a_{2n} and all the rest are equal to 0. In this section, whenever we use this notation, it will have this exact meaning.

Proof. The fact that $\mathcal{S}_{\alpha,\,\mathrm{even}}^\alpha(\mathbb{R}^d)$ is a closed subspace of $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ is trivial. It is a (DFS)-space as a closed subspace of a (DFS)-space. If $\psi = \sum_{n\in\mathbb{N}_0^d} a_nh_n \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, then $a_n = \int_{\mathbb{R}^d}\psi(x)h_n(x)dx$ and $\{a_n\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$ (cf. Proposition 1.6.1). Since $h_j(x)$ is even when j is even and is odd when j is odd, the last assertion in the proposition follows. $\square$

From now on, we fix $\alpha \geq 1$. The goal of this section is to give the explicit topological isomorphism between $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$.

Throughout this section, we denote by v and w the following mappings:

$$v : \mathbb{R}^d \to \overline{\mathbb{R}_+^d},\ v(x) = (x_1^2,\ldots,x_d^2),$$
$$w : \overline{\mathbb{R}_+^d} \to \overline{\mathbb{R}_+^d},\ w(x) = (\sqrt{x_1},\ldots,\sqrt{x_d}).$$

For $\gamma = (\gamma_1,\ldots,\gamma_d) \in \mathbb{R}^d$ such that $-\gamma_j \notin \mathbb{N}$, $j = 1,\ldots,d$ and $m \in \mathbb{N}_0^d$, we use the abbreviation

$$\binom{\gamma}{m} = \prod_{j=1}^d \binom{\gamma_j}{m_j}.$$

Moreover, we introduce the following notation $\mathbf{1/2} = (1/2,\ldots,1/2) \in \mathbb{R}_+^d$ and $\mathbf{3/2} = (3/2,\ldots,3/2) \in \mathbb{R}_+^d$.

Proposition 3.6.2. *Let $\phi = \sum_{n\in\mathbb{N}_0^d} a_n l_n$ be an element of $G_\alpha^\alpha(\mathbb{R}_+^d)$. Then $\phi\circ v$ is in $\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d)$ and*

$$\phi\circ v = \sum_{n\in\mathbb{N}_0^d} b_{2n}h_{2n},$$

where $\{b_{2n}\}_{n\in\mathbb{N}_0^d}\in s^\alpha$ is given by

$$b_{2n} = \frac{(-1)^{|n|}\pi^{d/4}\sqrt{(2n)!}}{2^{|n|}n!}\sum_{k\in\mathbb{N}_0^d} a_{k+n}\binom{k-\mathbf{1/2}}{k},\ n\in\mathbb{N}_0^d. \tag{3.28}$$

Moreover, the mapping

$$\phi\mapsto\phi\circ v,\ \ G_\alpha^\alpha(\mathbb{R}_+^d)\to\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d),$$

is a continuous injection.

Proof. By (1.12), for $n\in\mathbb{N}_0^d$ we have

$$L_n(x) = \sum_{m\le n}\binom{m-\mathbf{1/2}}{m}L_{n-m}^{-\mathbf{1/2}}(x),\ x\in\overline{\mathbb{R}_+^d}.$$

By (1.14),

$$L_j^{-1/2}(t^2) = \frac{(-1)^j}{2^{2j}j!}H_{2j}(t),\ \ t\in\mathbb{R},\ j\in\mathbb{N}_0.$$

Thus, for $x\in\mathbb{R}^d$, $n\in\mathbb{N}_0^d$,

$$\begin{aligned} l_n(v(x)) &= \pi^{d/4}\sum_{m\le n}\binom{m-\mathbf{1/2}}{m}\frac{(-1)^{|n-m|}\sqrt{(2n-2m)!}}{2^{|n-m|}(n-m)!}h_{2(n-m)}(x)\\ &= \pi^{d/4}\sum_{m\le n}\binom{n-m-\mathbf{1/2}}{n-m}\frac{(-1)^{|m|}\sqrt{(2m)!}}{2^{|m|}m!}h_{2m}(x). \end{aligned} \tag{3.29}$$

Let $\psi(x)=\phi(v(x))$, $x\in\mathbb{R}^d$. Clearly, $\psi\in\mathcal{C}(\mathbb{R}^d)$. Observe that,

$$\begin{aligned} \psi(x)=\phi(v(x)) &= \sum_{n\in\mathbb{N}_0^d} a_n l_n(v(x))\\ &= \pi^{d/4}\sum_{n\in\mathbb{N}_0^d} a_n\sum_{m\le n}\binom{n-m-\mathbf{1/2}}{n-m}\frac{(-1)^{|m|}\sqrt{(2m)!}}{2^{|m|}m!}h_{2m}(x). \end{aligned}$$

We will prove that the double series is absolutely convergent in $L^\infty(\mathbb{R}^d)$. By (1.16), we have $|h_n(x)|\le 1$, for all $n\in\mathbb{N}_0^d$, $x\in\mathbb{R}^d$. For $j\in\mathbb{N}$, we have

$$\binom{j-1/2}{j} = \frac{(2j-1)!!}{2^j j!}\le\frac{(2j)!!}{2^j j!} = 1. \tag{3.30}$$

This inequality trivially holds for $j = 0$ since, in this case, the left hand side is equal to 1. Hence,

$$\left| a_n \binom{n-m-\mathbf{1/2}}{n-m} \frac{(-1)^{|m|}\sqrt{(2m)!}}{2^{|m|}m!} h_{2m}(x) \right| \leq |a_n|,\ x \in \mathbb{R}^d,\ n \geq m.$$

Since $\{a_n\}_{n\in\mathbb{N}_0^d}$ is in s^α (cf. Theorem 3.3.1), the double series in the equality for $\psi(x)$ converges absolutely in $L^\infty(\mathbb{R}^d)$. Thus, we can change the order of summation in order to obtain

$$\begin{aligned} \psi(x) &= \pi^{d/4} \sum_{m\in\mathbb{N}_0^d} \frac{(-1)^{|m|}\sqrt{(2m)!}}{2^{|m|}m!} h_{2m}(x) \sum_{n\geq m} a_n \binom{n-m-\mathbf{1/2}}{n-m} \\ &= \sum_{m\in\mathbb{N}_0^d} b_{2m} h_{2m}(x), \end{aligned}$$

where

$$b_{2m} = \frac{(-1)^{|m|}\pi^{d/4}\sqrt{(2m)!}}{2^{|m|}m!} \sum_{n\in\mathbb{N}_0^d} a_{n+m} \binom{n-\mathbf{1/2}}{n},\ m \in \mathbb{N}_0^d.$$

If ϕ varies in a bounded subset B of $G_\alpha^\alpha(\mathbb{R}_+^d)$, then the sequence $\{a_n\}_{n\in\mathbb{N}_0^d}$ varies in a bounded subset of s^α (cf. Theorem 3.4.1). Since s^α is a (DFS)-space there exist $C, a > 1$ such that $|a_n| \leq Ca^{-|n|^{1/\alpha}}$, $\forall n \in \mathbb{N}_0^d$. The Cauchy-Schwarz inequality yields

$$|n|^{1/\alpha} + |m|^{1/\alpha} \leq 2(|n| + |m|)^{1/\alpha},\ \forall n, m \in \mathbb{N}_0^d. \tag{3.31}$$

Thus,

$$a^{-|n+m|^{1/\alpha}} \leq \sqrt{a}^{-|n|^{1/\alpha}} \sqrt{a}^{-|m|^{1/\alpha}}.$$

Hence, there exist $a', C' > 1$ such that

$$|a_{n+m}| \leq C' a'^{-|n|^{1/\alpha}} a'^{-|m|^{1/\alpha}}.$$

Using (3.30), we can estimate b_{2m} as follows

$$|b_{2m}| \leq C' a'^{-|m|^{1/\alpha}} \sum_{n\in\mathbb{N}_0^d} a'^{-|n|^{1/\alpha}} \leq C'' a''^{-|2m|^{1/\alpha}},\ m \in \mathbb{N}_0^d,$$

where $a'' = a'^{1/2^{1/\alpha}}$. Hence, when ϕ varies in B, the sequence $\{b_{2m}\}_{m\in\mathbb{N}_0^d}$ varies in a bounded subset of s^α. Thus, the mapping

$$\phi \mapsto \phi \circ v,\ \ G_\alpha^\alpha(\mathbb{R}_+^d) \to \mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d),$$

is well defined and it maps bounded sets into bounded sets (cf. Proposition 1.6.1 and Proposition 3.6.1). As $G_\alpha^\alpha(\mathbb{R}_+^d)$ is bornological, the mapping is continuous. Clearly, this mapping is injective. $\square$

Proposition 3.6.3. *Let* $\psi = \sum_{n\in\mathbb{N}_0^d} a_{2n}h_{2n} \in \mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d)$. *Then,* $\psi_{|\mathbb{R}^d_+} \circ w \in G^\alpha_\alpha(\mathbb{R}^d_+)$ *and*

$$\psi_{|\mathbb{R}^d_+} \circ w = \sum_{n\in\mathbb{N}_0^d} b_n l_n,$$

where $\{b_n\}_{n\in\mathbb{N}_0^d} \in s^\alpha$ *is given by*

$$b_n = \frac{(-1)^{|n|}2^{|n|}}{\pi^{d/4}} \sum_{k\in\mathbb{N}_0^d} \binom{k-\mathbf{3/2}}{k} \frac{(-1)^{|k|}2^{|k|}(k+n)!a_{2k+2n}}{\sqrt{(2k+2n)!}},\ n\in\mathbb{N}_0^d. \tag{3.32}$$

Moreover, the mapping

$$\psi \mapsto \psi_{|\mathbb{R}^d_+} \circ w,\ \ \mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d) \to G^\alpha_\alpha(\mathbb{R}^d_+),$$

is a continuous injection.

Proof. We represent h_{2n} through the finite Laguerre series. From (1.14), follows

$$H_{2n}(x) = (-1)^{|n|}2^{2|n|}n!L_n^{-\mathbf{1/2}}(v(x)),\ x\in\mathbb{R}^d,\ n\in\mathbb{N}_0^d.$$

Thus, by using (1.12), we have

$$\begin{aligned} H_{2n}(x) &= (-1)^{|n|}2^{2|n|}n!\sum_{m\le n}\binom{n-m-\mathbf{3/2}}{n-m} L_m(v(x)),\ x\in\mathbb{R}^d,\ n\in\mathbb{N}_0^d,\\ h_{2n}(w(x)) &= \frac{(-1)^{|n|}}{\pi^{d/4}}\sqrt{\frac{2^{2|n|}n!^2}{(2n)!}}\sum_{m\le n}\binom{n-m-\mathbf{3/2}}{n-m} l_m(x), \end{aligned} \tag{3.33}$$

$x\in\overline{\mathbb{R}^d_+}$ and $n\in\mathbb{N}_0^d$. Let $\psi = \sum_{n\in\mathbb{N}_0^d} a_{2n}h_{2n} \in \mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d)$. Then $\{a_{2n}\}_{n\in\mathbb{N}_0^d} \in s^\alpha$ (cf. Proposition 3.6.1). Hence, there exist $C, a > 1$ such that

$$|a_{2n}| \le Ca^{-|2n|^{1/\alpha}},\ \forall n\in\mathbb{N}_0^d. \tag{3.34}$$

Let $\phi(x) = \psi(w(x))$, $x\in\overline{\mathbb{R}^d_+}$. Clearly, $\phi\in\mathcal{C}(\overline{\mathbb{R}^d_+})$. We have

$$\phi(x) = \sum_{n\in\mathbb{N}_0^d} \frac{(-1)^{|n|}a_{2n}}{\pi^{d/4}}\sqrt{\frac{2^{2|n|}n!^2}{(2n)!}}\sum_{m\le n}\binom{n-m-\mathbf{3/2}}{n-m} l_m(x). \tag{3.35}$$

By (1.5), $|l_n(x)|\le 1$, for all $x\in\overline{\mathbb{R}^d_+}$, $n\in\mathbb{N}_0^d$. Similarly as in (3.30), we have

$$\left|\binom{n-m-\mathbf{3/2}}{n-m}\right| \le 1,\ \text{for all } n\ge m, n, m\in\mathbb{N}^d.$$

Since,

$$\binom{2j}{j} \sim \frac{4^j}{\sqrt{j\pi}},\ \text{as } j\to\infty,$$

for some $C_1 > 1$, we obtain

$$\left| \frac{(-1)^{|n|}}{\pi^{d/4}} \sqrt{\frac{2^{2|n|} n!^2}{(2n)!}} \binom{n-m-\mathbf{3/2}}{n-m} l_m(x) \right| \leq C_1(|n|+1)^{d/2}, \tag{3.36}$$

where $n, m \in \mathbb{N}_0^d$, $n \geq m$. By using (3.34), we can conclude that the series on the right hand side in (3.35) converges absolutely in $L^\infty(\overline{\mathbb{R}_+^d})$. Thus, we can change the order of summation in order to obtain $\phi(x) = \sum_{m\in\mathbb{N}_0^d} b_m l_m(x)$, where

$$b_m = \frac{(-1)^{|m|}2^{|m|}}{\pi^{d/4}} \sum_{n\in\mathbb{N}_0^d} \binom{n-\mathbf{3/2}}{n} \frac{(-1)^{|n|}2^{|n|}(n+m)! a_{2n+2m}}{\sqrt{(2n+2m)!}}.$$

To estimate b_m we can perform analogous technique as for (3.36). Hence, we obtain

$$|b_m| \leq C_2 \sum_{n\in\mathbb{N}_0^d} (|n+m|+1)^{d/2} a^{-|2n+2m|^{1/\alpha}} \leq C_3 \sum_{n\in\mathbb{N}_0^d} a'^{-|n+m|^{1/\alpha}}, \ \forall m \in \mathbb{N}_0^d,$$

for some $1 < a' < a$. Now, (3.31) implies that there exist $C'', a'' > 1$ such that

$$|b_m| \leq C'' a''^{-|m|^{1/\alpha}}, \ \ \forall m \in \mathbb{N}_0^d,$$

i.e. $\{b_m\}_{m\in\mathbb{N}_0^d} \in s^\alpha$. Thus, $\phi \in G_\alpha^\alpha(\mathbb{R}_+^d)$. If ψ varies in a bounded subset B of $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$, then (3.34) holds with the same $C, a > 1$ for all the sequences $\{a_{2n}\}_{n\in\mathbb{N}_0^d}$ generated by $\psi \in B$ (since $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$ is a subspace of a (DFS)-space $\mathcal{S}_{\alpha/2}^{\alpha/2}(\mathbb{R}^d)$). Thus, from the above proof it follows that $\{b_m\}_{m\in\mathbb{N}_0^d}$ varies in a bounded subset of s^α, i.e. ϕ varies in a bounded subset of $G_\alpha^\alpha(\mathbb{R}_+^d)$. Hence, the mapping

$$\psi \mapsto \psi_{\mathbb{R}_+^d} \circ w, \ \ \mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d) \to G_\alpha^\alpha(\mathbb{R}_+^d),$$

is well defined and maps bounded sets into bounded sets. As $\mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$ is a (DFS)-space (cf. Proposition 3.6.1), it is bornological. Hence, the mapping is continuous. The proof for the injectivity is trivial. □

Combining the above two propositions, we obtain the following result.

Theorem 3.6.1. *The mapping*

$$\phi \mapsto \phi \circ v, \ \ G_\alpha^\alpha(\mathbb{R}_+^d) \to \mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d)$$

is a topological isomorphism. If $\phi = \sum_{n\in\mathbb{N}_0^d} a_n l_n$, *then* $\phi \circ v = \sum_{n\in\mathbb{N}_0^d} b_{2n} h_{2n}$, *where* $\{b_{2n}\}_{n\in\mathbb{N}_0^d} \in s^\alpha$ *is given by (3.28).*

The inverse of this mapping is given by

$$\psi \mapsto \psi_{|\mathbb{R}_+^d} \circ w, \ \ \mathcal{S}_{\alpha/2,\,\mathrm{even}}^{\alpha/2}(\mathbb{R}^d) \to G_\alpha^\alpha(\mathbb{R}_+^d).$$

If $\psi = \sum_{n\in\mathbb{N}_0^d} a_{2n} h_{2n}$, *then* $\psi \circ w = \sum_{n\in\mathbb{N}_0^d} b_n l_n$, *where* $\{b_n\}_{n\in\mathbb{N}_0^d} \in s^\alpha$ *is given by (3.32).*

For the moment, we denote by X the subspace of $(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'$ consisting of all $T \in (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'$ such that $T = \sum_{n\in\mathbb{N}^d_0} a_{2n}h_{2n}$ for some $\{a_{2n}\}_{n\in\mathbb{N}^d_0} \in (s^{2\alpha})'$. Of course, these are exactly the "even" tempered ultradistributions, i.e. the elements of $(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'$ which remain unchanged under the antipode mappings in each coordinate (cf. (3.27)). It is easy to verify that X is closed subspace, hence, it is an (FS)-space.

Proposition 3.6.4. *The strong dual of $\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)$ is topologically isomorphic to X.*

Proof. By Proposition 3.6.1, $\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)$ is a (DFS)-space which is a closed subspace of the (DFS)-space $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d)$, hence, Theorem 2.4.4 implies that the strong dual $(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))'$ of $\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)$ is topologically isomorphic to the (FS)-space $(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'/(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp}$ where

$$(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp} = \{T \in (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))' |\, \langle T, \psi\rangle = 0,\ \forall \psi \in \mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)\}$$

is the orthogonal space to $\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)$. Denoting by $\hat{T} \in (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'/(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp}$ the coset of $T \in (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'$, we define the mapping

$$I : X \to (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'/(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp},$$

$I(T) = \hat{T}$. It is easy to verify that I is injective. For $\hat{T} \in (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'/(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp}$ let $T = \sum_n b_n h_n$. Then $T_1 = \sum_n b_{2n}h_{2n} \in X$ and $T - T_1 \in (\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp}$. Hence, $I(T_1) = \hat{T}$, which proves the surjectivity of I. Moreover, I is continuous since it decomposes as

$$X \to (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))' \to (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'/(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp},$$

where the first mapping is the canonical injection and the second is the natural mapping. Since X and $(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'/(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))^{\perp}$ are (F)-spaces, the open mapping theorem proves that I is topological isomorphism. $\square$

Remark 3.6.2. Until the end of this section, we will identify $(\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))'$ (the strong dual of $\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)$) with X. It follows directly from the proof that each $T \in (\mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))'$ can be represented as $\sum_{n\in\mathbb{N}^d_0} b_{2n}h_{2n}$, where $\{b_{2n}\}_{n\in\mathbb{N}^d_0} \in (s^{2\alpha})'$ and for $\psi = \sum_{n\in\mathbb{N}^d_0} a_{2n}h_{2n} \in \mathcal{S}^{\alpha}_{\alpha,\,\mathrm{even}}(\mathbb{R}^d)$, we have $\langle T, \psi\rangle = \sum_{n\in\mathbb{N}^d_0} a_{2n}b_{2n}$.

If we denote by $\mathfrak{I}$ the isomorphism

$$\psi \mapsto \psi_{|\mathbb{R}^d_+} \circ w,\ \mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d) \to G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$$

and by $\mathfrak{I}^{-1}$ its inverse

$$\mathfrak{I}^{-1} : \phi \mapsto \phi \circ v,\ G^{\alpha}_{\alpha}(\mathbb{R}^d_+) \to \mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d),$$

then the transpose ${}^t\mathfrak{I}$ is an isomorphism between $(G^{\alpha}_{\alpha}(\mathbb{R}^d_+))'$ and $(\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d))'$.

By Proposition 3.6.4 (and Remark 3.6.2), for $T = \sum_n a_n l_n \in (G^{\alpha}_{\alpha}(\mathbb{R}^d_+))'$ there exists $\{b_{2n}\}_{n\in\mathbb{N}^d_0} \in (s^{\alpha})'$ such that

$${}^t\mathfrak{I}T = \sum_n b_{2n}h_{2n} \in (S^{\alpha}_{\alpha}(\mathbb{R}^d))'.$$

Then, (3.33) implies

$$b_{2n} = \langle {}^t\mathfrak{I}T, h_{2n}\rangle = \frac{(-1)^{|n|}2^{|n|}n!}{\pi^{d/4}\sqrt{(2n)!}} \sum_{m\le n} \binom{n-m-\mathbf{3/2}}{n-m} a_m. \tag{3.37}$$

Similarly, given $T = \sum_n a_{2n}h_{2n} \in (S^\alpha_{\alpha,\,\mathrm{even}}(\mathbb{R}^d))'$, ${}^t(\mathfrak{I}^{-1})T \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$. Hence,

$${}^t(\mathfrak{I}^{-1})T = \sum_n b_n l_n,$$

for some $\{b_n\}_{n\in\mathbb{N}_0^d} \in (s^\alpha)'$. The equality (3.29) implies

$$b_n = \langle {}^t(\mathfrak{I}^{-1})T, l_n\rangle = \pi^{d/4} \sum_{m\le n} \binom{n-m-\mathbf{1/2}}{n-m} \frac{(-1)^{|m|}\sqrt{(2m)!}}{2^{|m|}m!} a_{2m}. \tag{3.38}$$

Since ${}^t(\mathfrak{I}^{-1}) = ({}^t\mathfrak{I})^{-1}$, we have proved the following theorem.

Theorem 3.6.2. *The transpose ${}^t\mathfrak{I}$ of the isomorphism*

$$\mathfrak{I} : \psi \mapsto \psi_{|\mathbb{R}^d_+} \circ w,\ \mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d) \to G^\alpha_\alpha(\mathbb{R}^d_+),$$

is a topological isomorphism

$${}^t\mathfrak{I} : (G^\alpha_\alpha(\mathbb{R}^d_+)) \mapsto (\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d))'.$$

The image of $\sum_n a_n l_n \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$ under this isomorphism is $\sum_n b_{2n}h_{2n}$, where $\{b_{2n}\}_{n\in\mathbb{N}_0^d} \in (s^\alpha)'$ is given by (3.37).

The inverse of this isomorphism, $({}^t\mathfrak{I})^{-1}$ maps $\sum_n a_{2n}h_{2n} \in (\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d))'$ to $\sum_n b_n l_n \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$, where $\{b_n\}_{n\in\mathbb{N}_0^d} \in (s^\alpha)'$ is given by (3.38).

3.7 Structural theorems for $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, $\alpha \ge 1$

In this section we state two structural theorems for $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, $\alpha \ge 1$.

3.7.1 The first structural theorem

For the terminology used in this subsection, we refer to Appendix B.

Remark 3.7.1. We will need the following estimate in the sequel

$$\sum_{j=0}^\infty \frac{s^j}{j!^\alpha} \le e^{\alpha s^{1/\alpha}},\ s \ge 0,\ \alpha \ge 1. \tag{3.39}$$

Moreover,

$$\sup_{j\in\mathbb{N}_0} \frac{s^j}{j!^\alpha} = \left(\sup_{j\in\mathbb{N}_0} \frac{s^{j/\alpha}}{j!}\right)^\alpha \ge \left(\frac{1}{2}\sum_{j=0}^\infty \frac{(s^{1/\alpha})^j}{2^j j!}\right)^\alpha = 2^{-\alpha}e^{(\alpha/2)s^{1/\alpha}},\ s \ge 0,\ \alpha \ge 1,$$

i.e. there exists $c > 0$ such that

$$\sup_{j\in\mathbb{N}_0} \frac{s^j}{j!^\alpha} \ge ce^{cs^{1/\alpha}},\ s \ge 0,\ \alpha \ge 1. \tag{3.40}$$

Before we state the next result, notice that the operator

$$R^k = \prod_{j=1}^{d} \left(x_j D_{x_j}^2 + D_{x_j} - \frac{x_j}{4} + \frac{1}{2} \right)^{k_j}, \quad k \in \mathbb{N}_0^d. \tag{3.41}$$

is continuous on $\mathcal{S}(\mathbb{R}_+^d)$ and on $\mathcal{S}'(\mathbb{R}_+^d)$ (recall (1.4) for the definition of R).

Lemma 3.7.1. *For each $k \in \mathbb{N}_0^d$, R^k acts continuously on $G_\alpha^\alpha(\mathbb{R}_+^d)$.*

Proof. If $\phi = \sum_{n\in\mathbb{N}_0^d} a_n l_n$ varies in a bounded subset of $G_\alpha^\alpha(\mathbb{R}_+^d)$, then $\{a_n\}_{n\in\mathbb{N}_0^d}$ varies in a bounded subset of s^α. Since $\sum_{n\in\mathbb{N}_0^d} a_n l_n$ converges absolutely to ϕ in $\mathcal{S}(\mathbb{R}_+^d)$, we have

$$R^k\phi = \sum_n a_n R^k l_n = \sum_n a_n (-1)^{|k|} n^k l_n$$

and the series converges absolutely in $\mathcal{S}(\mathbb{R}_+^d)$.

It can be easily proved that $\{a_n(-1)^{|k|}n^k\}_{n\in\mathbb{N}_0^d}$ is in s^α and when $\{a_n\}_{n\in\mathbb{N}_0^d}$ varies in a bounded subset of s^α so does $\{a_n(-1)^{|k|}n^k\}_{n\in\mathbb{N}_0^d}$. Hence, R^k is well defined as a mapping from $G_\alpha^\alpha(\mathbb{R}_+^d)$ onto itself and it maps bounded sets into bounded sets. As $G_\alpha^\alpha(\mathbb{R}_+^d)$ is bornological, R^k is continuous. □

By duality, we can define the transpose ${}^tR^k$ of R^k as a continuous operator on $(G_\alpha^\alpha(\mathbb{R}_+^d))'$. If $T = \sum_{n\in\mathbb{N}_0^d} b_n l_n$, then one easily verifies that

$${}^tR^k T = \sum_n b_n (-1)^{|k|} n^k l_n$$

(since $\{b_n\}_{n\in\mathbb{N}_0^d} \in (s^\alpha)'$, the sequence $\{b_n(-1)^{|k|}n^k\}_{n\in\mathbb{N}_0^d}$ also belongs to $(s^\alpha)'$ and thus the right hand side is a well defined element of $(G_\alpha^\alpha(\mathbb{R}_+^d))'$). We come to the conclusion that ${}^tR^k$ coincides with R^k when $T \in G_\alpha^\alpha(\mathbb{R}^d) \subseteq (G_\alpha^\alpha(\mathbb{R}_+^d))'$. Hence, from now on, we will write R^k instead of ${}^tR^k$.

By Remark 3.7.1 and Proposition B.0.1, $P(z) = \sum_n c_n z^n$ is an ultrapolynomial of class $\{p!^\alpha\}$ if and only if for every $h > 0$ there exists $C > 0$ such that

$$|P(z)| \le C e^{h|z|^{1/\alpha}}, \ \forall z \in \mathbb{C}^d.$$

Next, for a given ultrapolynomial $P(z) = \sum_n c_n z^n$ of class $\{p!^\alpha\}$, we will show that the operator $\sum_n c_n R^n$, denoted by $P(R)$, is a well defined and continuous operator on both $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $(G_\alpha^\alpha(\mathbb{R}_+^d))'$. In the proof we will use the fact that $\mathcal{L}_b(G_\alpha^\alpha(\mathbb{R}_+^d), G_\alpha^\alpha(\mathbb{R}_+^d))$ and $\mathcal{L}_b((G_\alpha^\alpha(\mathbb{R}_+^d))', (G_\alpha^\alpha(\mathbb{R}_+^d))')$ are complete (cf. Corollary A.4.1; notice that $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $(G_\alpha^\alpha(\mathbb{R}_+^d))'$ are bornological and complete spaces).

Lemma 3.7.2. *Let*

$$P(z) = \sum_{n\in\mathbb{N}_0^d} c_n z^n$$

be an ultrapolynomial of class $\{p!^\alpha\}$. Then,

$$\sum_{n\in\mathbb{N}_0^d} c_n R^n$$

converges absolutely in both $\mathcal{L}_b(G_\alpha^\alpha(\mathbb{R}_+^d), G_\alpha^\alpha(\mathbb{R}_+^d))$ and $\mathcal{L}_b((G_\alpha^\alpha(\mathbb{R}_+^d))', (G_\alpha^\alpha(\mathbb{R}_+^d))')$.

Proof. Since $G_\alpha^\alpha(\mathbb{R}_+^d)$ is a barrelled and complete space, its topology is given by the system of seminorms

$$\phi \mapsto \sup_{T\in B'} |\langle T, \phi\rangle|,$$

where B' ranges over all bounded subsets of $(G_\alpha^\alpha(\mathbb{R}_+^d))'$. Hence, the topology of $\mathcal{L}_b(G_\alpha^\alpha(\mathbb{R}_+^d), G_\alpha^\alpha(\mathbb{R}_+^d))$ is given by the system of seminorms

$$\Phi \mapsto \sup_{\substack{T\in B'\\ \phi\in B}} |\langle T, \Phi(\phi)\rangle|,$$

where B and B' range over all bounded subsets of $G_\alpha^\alpha(\mathbb{R}_+^d)$ and $(G_\alpha^\alpha(\mathbb{R}_+^d))'$, respectively. To prove that $\sum_{n\in\mathbb{N}_0^d} c_n R^n$ converges absolutely in $\mathcal{L}_b(G_\alpha^\alpha(\mathbb{R}_+^d), G_\alpha^\alpha(\mathbb{R}_+^d))$, we have to prove that for each such B and B',

$$\sum_{n\in\mathbb{N}_0^d} |c_n| \sup_{\substack{T\in B'\\ \phi\in B}} |\langle T, R^n\phi\rangle| < \infty. \tag{3.42}$$

Now, fix such B and B'. Let

$$\phi = \sum_n a_{n,\phi} l_n, \quad \phi \in B$$

and

$$T = \sum_n b_{n,T} l_n, \quad T \in B'.$$

Thus, $\{\{a_{n,\phi}\}_n | \, \phi \in B\}$ is bounded in s^α and $\{\{b_{n,T}\}_n | T \in B'\}$ is bounded in $(s^\alpha)'$. There exist $a, C > 1$ such that

$$|a_{n,\phi}| \leq C a^{-|n|^{1/\alpha}}, \text{ for all } n \in \mathbb{N}_0^d,\ \phi \in B.$$

For this a, choose $1 < b \leq a^{1/4}$. Then, there exists $C_1 > 0$ such that

$$|b_{n,T}| \leq C_1 b^{|n|^{1/\alpha}} \text{, for all } n \in \mathbb{N}_0^d,\ T \in B'.$$

Moreover, there exist $s, C_2 > 1$ such that

$$|m|^{|n|} \leq C_2 s^{|n|} b^{|m|^{1/\alpha}} |n|!^\alpha, \text{ for all } n, m \in \mathbb{N}_0^d.$$

Hence,

$$\sup_{\substack{T\in B'\\ \phi\in B}} |\langle T, R^n\phi\rangle| \leq \sup_{\substack{T\in B'\\ \phi\in B}} \sum_{m\in\mathbb{N}_0^d} |a_{m,\phi}||b_{m,T}||m|^{|n|} \leq C_3 s^{|n|}|n|!^\alpha,\ \forall n \in \mathbb{N}_0^d.$$

Since P is an ultrapolynomial of class $\{p!^\alpha\}$, the last inequality implies (3.42).

The topology of $\mathcal{L}_b((G_\alpha^\alpha(\mathbb{R}_+^d))', (G_\alpha^\alpha(\mathbb{R}_+^d))')$ is given by the system of seminorms

$$\Phi \mapsto \sup_{\substack{T\in B'\\ \phi\in B}} |\langle \Phi(T), \phi\rangle|,$$

where B and B' range over all bounded subsets of $G^\alpha_\alpha(\mathbb{R}^d_+)$ and $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, respectively. To prove that $\sum_{n\in\mathbb{N}^d_0} c_n R^n$ converges absolutely in $\mathcal{L}_b((G^\alpha_\alpha(\mathbb{R}^d_+))', (G^\alpha_\alpha(\mathbb{R}^d_+))')$ we need to prove that for each such B and B'

$$\sum_{n\in\mathbb{N}^d_0} |c_n| \sup_{\substack{T\in B'\\ \phi\in B}} |\langle R^n T, \phi\rangle| < \infty.$$

This can be done by the same technique as above. □

Before we prove the main result of this subsection, we state the following three technical lemmas. The first one is proved in [34].

Lemma 3.7.3. *([34, Lemma 2.4]) Let $g:[0,\infty)\to[0,\infty)$ be an increasing function such that satisfies the following estimate: for every $h>0$ there exists $C>0$ such that $g(t)\le M(ht)+\ln C$. Then there exists a subordinate function $\epsilon(t)$ such that $g(t)\le M(\epsilon(t))+\ln C'$, for some constant $C'>1$.*

For the definition of a subordinate function see Appendix B, Definition B.0.1.

Lemma 3.7.4. *Let B be a bounded subset of $(s^\alpha)'$. There exists a sequence $\{r_p\}_{p\in\mathbb{N}}$ of positive numbers such that increases monotonically to infinity and $C'>1$ such that*

$$|b_n| \le C' e^{N_{r_p}(|n|)}, \quad \text{for all } n\in\mathbb{N}^d_0,\ \{b_n\}_{n\in\mathbb{N}^d_0}\in B.$$

Proof. Since B is a bounded subset of $(s^\alpha)'$, for every $h>0$ there exists $C>1$ such that

$$|b_n| \le C e^{M(h|n|)}, \quad \text{for all } n\in\mathbb{N}^d_0,\ \{b_n\}_n\in B'$$

(cf. Remark 3.7.1). Define $f:[0,\infty)\to[0,\infty)$ as

$$f(t) = \sup_{\substack{|k|\le t\\ \{b_n\}_n\in B}} \ln_+ |b_k|, \ t\in[0,\infty).$$

One easily verifies that f is a nonnegative monotonically increasing function and for every $h>0$ there exists $C>0$ such that

$$f(t)\le M(ht)+C.$$

Thus, we can apply Lemma 3.7.3 to obtain the existence of a subordinate function $\epsilon:[0,\infty)\to[0,\infty)$ and $C_1>1$ such that

$$f(t)\le M(\epsilon(t))+C_1, \quad t\in[0,\infty).$$

Now, Lemma B.0.1 implies the existence of a sequence N_p, $p\in\mathbb{N}_0$, of positive numbers which satisfies $(M.1)$ such that

$$M(\epsilon(t))\le N(t), \quad t\in(0,\infty)$$

($N(\cdot)$ is the associated function of the sequence N_p) and

$$\frac{N_p M_{p-1}}{N_{p-1}M_p}\to\infty, \quad \text{as } p\to\infty.$$

Define

$$r'_p = \frac{N_p M_{p-1}}{N_{p-1} M_p}, \quad p \in \mathbb{N}.$$

Since $r'_p \to \infty$, one can find a monotonically increasing sequence of positive numbers $\{r_p\}_{p\in\mathbb{N}_0}$ which tends to infinity and $r_p \le r'_p$, $p \in \mathbb{N}$. Then,

$$\begin{aligned} f(t) &\le N(t) + C_1 = \sup_{p\in\mathbb{N}_0} \ln \frac{t^p N_0}{N_p} + C_1 = \sup_{p\in\mathbb{N}_0} \ln \frac{t^p}{M_p \prod_{j=1}^p r'_j} + C_1 \\ &\le \sup_{p\in\mathbb{N}_0} \ln \frac{t^p}{M_p \prod_{j=1}^p r_j} + C_1 = N_{r_p}(t) + C_1. \end{aligned}$$

By the definition of f, this readily implies the conclusion of the lemma. $\square$

Let $\{r_p\}_{p\in\mathbb{N}}$ be a sequence of positive numbers such that increases monotonically to infinity. For any $\{r_p\}_{p\in\mathbb{N}}$, we can construct a new sequence such that the zeroth term of the sequence is equal to $0!^\alpha = 1$ and the p-th term is equal to $p!^\alpha \prod_{j=1}^p r_j$, $p \in \mathbb{N}$. This sequence also satisfies the condition $(M.1)$ (see Appendix B) and one can define its associated function which we denote by $N_{r_p}(\cdot)$. The next lemma is proved in [33]; here $\mathfrak{R}$ stands for a set of all sequences of positive numbers such that increase monotonically to infinity.

Lemma 3.7.5. *([33, Lemma 2.1], Roumieu case) Let $r' \ge 1$ and $(k_p) \in \mathfrak{R}$. There exists an ultrapolynomial $P(z)$ of class $\{M_p\}$ such that P does not vanish on $\mathbb{R}^d$ and satisfies the following estimate:*

There exists $C > 0$ such that for all $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$,

$$|D^\alpha (1/P(x))| \le C \frac{\alpha!}{r'^{|\alpha|}} e^{-N_{k_p}(|x|)}.$$

As a special case, we see that for any given sequence of positive numbers $\{r_p\}_{p\in\mathbb{N}}$ such that increases monotonically to infinity, one can find an ultrapolynomial $P(z)$ of class $\{p!^\alpha\}$ and $C > 0$ such that

$$|P(x)| \ge C e^{N_{r_p}(|x|)} \text{ for all } x \in \mathbb{R}^d.$$

Theorem 3.7.1. *Let $B' \subseteq (G^\alpha_\alpha(\mathbb{R}^d_+))'$ be a bounded set. There exists an ultrapolynomial $P(z)$ of class $\{p!^\alpha\}$ and a bounded set B in $L^2(\mathbb{R}^d_+)$ such that for each $T \in B'$ there exists $F_T \in B$ satisfying*

$$T = P(R) F_T \in (G^\alpha_\alpha(\mathbb{R}^d_+))'.$$

Conversely, given a bounded set B in $L^2(\mathbb{R}^d_+)$ and an ultrapolynomial $P(z)$ of class $\{p!^\alpha\}$,

$$P(R)F \in (G^\alpha_\alpha(\mathbb{R}^d_+))', \quad \textit{for each } F \in B$$

and the set $\{P(R)F |\, F \in B'\}$ is bounded in $(G^\alpha_\alpha(\mathbb{R}^d_+))'$.

Proof. Let $T=\sum_{n\in\mathbb{N}_0^d} b_{n,T}l_n$, $T\in B'$. The set $\{\{b_{n,T}\}_{n\in\mathbb{N}_0^d}|\, T\in B'\}$ is bounded in $(s^\alpha)'$. Lemma 3.7.4 implies that there exists a sequence of positive numbers $\{r_p\}_{p\in\mathbb{N}}$ such that not only increases monotonically to infinite but also

$$|b_{n,T}|\le C' e^{N_{r_p}(|n|)},\ \text{for all } n\in\mathbb{N}_0^d,\ T\in B'.$$

We define the sequence $\{r'_p\}_{p\in\mathbb{N}}$ by

$$r'_j=\min\{1,r_1\},\ j=1,\dots,d+1$$

and

$$r'_j=r_{j-d-1},\ j\ge d+2,\ j\in\mathbb{N}.$$

Then, $\{r'_p\}_{p\in\mathbb{N}}$ increases monotonically to infinity, $r'_p\le r_p$, $p\in\mathbb{N}$ and there exists $\tilde{C}_1\ge 1$ such that

$$(t^{d+1}+1)e^{N_{r_p}(t)}\le \tilde{C}_1 e^{N_{r'_p}(2^\alpha t)}+e^{N_{r_p}(t)},\ t\in[0,\infty).$$

Hence, if we define $k_p=r'_p/2^\alpha$, $p\in\mathbb{N}$, the sequence $\{k_p\}_{p\in\mathbb{N}}$ increases monotonically to infinity and there exists $\tilde{C}_2>1$ such that

$$(t^{d+1}+1)e^{N_{r_p}(t)}\le \tilde{C}_2 e^{N_{k_p}(t)},\ t\in[0,\infty).$$

By Lemma 3.7.5, we can choose an ultrapolynomial $P(z)=\sum_{n\in\mathbb{N}_0^d} c_n z^n$ of class $\{p!^\alpha\}$ such that

$$|P(x)|\ge Ce^{N_{k_p}(|x|)},\ \text{for all } x\in\mathbb{R}^d.$$

Lemma 3.7.2 verifies that $P(R)$ acts continuously on $G_\alpha^\alpha(\mathbb{R}_+^d)$ and on $(G_\alpha^\alpha(\mathbb{R}_+^d))'$. Observe that

$$\sum_{n\in\mathbb{N}_0^d}\left|\frac{b_{n,T}}{P(-n)}\right|^2\le C_1\sum_{n\in\mathbb{N}_0^d} e^{2N_{r_p}(|n|)}e^{-2N_{k_p}(|n|)}\le C_2,\ \forall T\in B'.$$

Hence,

$$F_T=\sum_{n\in\mathbb{N}_0^d}\frac{b_{n,T}}{P(-n)}l_n\in L^2(\mathbb{R}_+^d)$$

and the set $\{F_T|\, T\in B'\}$ is bounded in $L^2(\mathbb{R}_+^d)$. As $L^2(\mathbb{R}_+^d)\subseteq(G_\alpha^\alpha(\mathbb{R}_+^d))'$, $P(R)F_T\in(G_\alpha^\alpha(\mathbb{R}_+^d))'$. Moreover,

$$P(R)l_n=\sum_{m\in\mathbb{N}_0^d} c_m R^m l_n=\sum_{n\in\mathbb{N}_0^d} c_m(-n)^m l_n=P(-n)l_n.$$

Hence,

$$P(R)F_T=\sum_{n\in\mathbb{N}_0^d}\frac{b_{n,T}}{P(-n)}P(R)l_n=\sum_{n\in\mathbb{N}_0^d} b_{n,T}l_n=T.$$

The converse part of the theorem is trivial. $\square$

3.7.2 The second structural theorem

Remark 3.1.1 will enable us to prove the results of this subsection.

Proposition 3.7.1. *Let $A > 0$. For each $T \in (\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))'$, there exists $j \in \mathbb{N}_0$ and $F_{A,p,k} \in L^2(\mathbb{R}^d_+)$, $p,k \in \mathbb{N}^d_0$ and $\tilde{F}_{A,n,m} \in L^2(\mathbb{R}^d_+)$, $n,m \in \mathbb{N}^d_0$ with $|n| \leq j$, $|m| \leq j$ such that*

$$\sum_{p,k\in\mathbb{N}^d_0} A^{2|p+k|}p^{\alpha p}k^{\alpha k}\|F_{A,p,k}\|^2_{L^2(\mathbb{R}^d_+)} + \sum_{|m|\leq j,\,|n|\leq j} \|\tilde{F}_{A,n,m}\|^2_{L^2(\mathbb{R}^d_+)} < \infty \tag{3.43}$$

and for all $\phi \in \tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$

$$\begin{aligned} \langle T,\phi\rangle &= \sum_{p,k\in\mathbb{N}^d_0} \int_{\mathbb{R}^d_+} F_{A,p,k}(x)x^{(p+k)/2}D^p\phi(x)dx \\ &\quad + \sum_{|m|\leq j,\,|n|\leq j} \int_{\mathbb{R}^d_+} \tilde{F}_{A,n,m}(x)x^m D^n\phi(x)dx. \end{aligned} \tag{3.44}$$

Conversely, given $j \in \mathbb{N}_0$ and the set of $L^2(\mathbb{R}^d_+)$-functions

$$\{F_{A,p,k}|\, p,k \in \mathbb{N}^d_0\} \cup \{\tilde{F}_{A,n,k}|\, n,m \in \mathbb{N}^d_0,\ |n| \leq j,\ |m| \leq j\}$$

such that (3.43) holds, there exists $T \in (\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))'$ given by (3.44).

Proof. For $j \in \mathbb{N}_0$, we define

$$\mathbf{U}_j = \bigsqcup_{(p,k)\in\mathbb{N}^{2d}_0} \mathbb{R}^d_{+,p,k} \sqcup \bigsqcup_{\substack{(n,m)\in\mathbb{N}^{2d}_0\\ |n|\leq j,\,|m|\leq j}} \mathbb{R}^d_{+,n,m},$$

where, as standard, $\bigsqcup$ denotes a disjoint union. The each member of this disjoint union is an exact copy of $\mathbb{R}^d_+$. We equip $\mathbf{U}_j$ with the disjoint union topology. Since there are countably many copies of $\mathbb{R}^d_+$, $\mathbf{U}_j$ is a Hausdorff locally compact space and an each open set in $\mathbf{U}_j$ is σ-compact. We define a Borel measure μ_j on $\mathbf{U}_j$ by

$$\mu_j(E) = \sum_{(p,k)\in\mathbb{N}^{2d}_0} A^{-2|p+k|}p^{-\alpha p}k^{-\alpha k}|E\cap\mathbb{R}^d_{+,p,k}| + \sum_{\substack{(n,m)\in\mathbb{N}^{2d}_0\\ |n|\leq j,\,|m|\leq j}} |E\cap\mathbb{R}^d_{+,n,m}|,$$

where $|E\cap\mathbb{R}^d_{+,p,k}|$ and $|E\cap\mathbb{R}^d_{+,n,m}|$ is the Lebesgue measure of $E\cap\mathbb{R}^d_{+,p,k}$ and $|E\cap\mathbb{R}^d_{+,n,m}|$, respectively. Note that E is a Borel set in $\mathbf{U}_j$ if and only if $E\cap\mathbb{R}^d_{+,p,k}$ and $E\cap\mathbb{R}^d_{+,n,m}$ are Borel sest in $\mathbb{R}^d_{+,p,k}$ and $\mathbb{R}^d_{+,n,m}$, respectively, for all $p,k,n,m \in \mathbb{N}^d_0$, $|m| \leq j$, $|n| \leq j$. As readily seen, μ_j is locally finite, σ-finite and $\mu_j(\mathbf{K}) < \infty$ for every compact set $\mathbf{K}$ in $\mathbf{U}_j$. By the properties of $\mathbf{U}_j$, μ_j is regular (both inner and outer regular). Now, observe that, for each $j \in \mathbb{N}_0$, $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ is continuously injected into $L^2(\mathbf{U}_j,\mu_j)$ by the mapping $\mathfrak{J}_j : \phi \mapsto \mathbf{F}$, where $\mathbf{F}$ is defined by

$$\mathbf{F}_{|\mathbb{R}^d_{+,p,k}} = x^{(p+k)/2}D^p\phi(x),\ p,k \in \mathbb{N}^d_0$$

and

$$\mathbf{F}_{|\mathbb{R}^d_+,n,m} = x^m D^n \phi(x),\ n,m \in \mathbb{N}_0^d,\ m| \leq j,\ |n| \leq j.$$

In fact,

$$\begin{aligned}(\tilde{\sigma}_{A,j}(\phi))^2 &= \sum_{p,k\in\mathbb{N}_0^d} \frac{\|x^{(p+k)/2} D^p \phi(x)\|^2_{L^2(\mathbb{R}^d_+)}}{A^{2|p+k|} k^{\alpha k} p^{\alpha p}} + \sum_{|m|\leq j,\,|n|\leq j} \|x^m D^n \phi(x)\|^2_{L^2(\mathbb{R}^d_+)} \\ &= \int_{\mathbf{U}_j} |\mathbf{F}|^2 d\mu_j = \|\mathbf{F}\|^2_{L^2(\mathbf{U}_j,\mu_j)}. \end{aligned} \tag{3.45}$$

If $T \in (\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))$, there exist $j \in \mathbb{N}_0$ and $C > 0$ such that $|\langle T, \phi\rangle| \leq C\tilde{\sigma}_{A,j}(\phi)$. Because of (3.45), T induces a continuous functional on $\mathfrak{J}_j(\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))$ when this space is equipped with the topology induced by $L^2(\mathbf{U}_j, \mu_j)$. By the Hahn-Banach theorem, we can extend T to a continuous functional $\mathbf{T}$ on the whole $L^2(\mathbf{U}_j, \mu_j)$ and hence $\mathbf{T} \in L^2(\mathbf{U}_j, \mu_j)$. Denote

$$F_{A,p,k} = A^{-2|p+k|} p^{-\alpha p} k^{-\alpha k} \mathbf{T}_{|\mathbb{R}^d_+,p,k},\ \tilde{F}_{A,n,m} = \mathbf{T}_{|\mathbb{R}^d_+,n,m},$$

where $p, k, n, m \in \mathbb{N}_0^d$, $|m| \leq j$, $|n| \leq j$. Then, $F_{A,p,k}, \tilde{F}_{A,n,m} \in L^2(\mathbb{R}^d_+)$, for all $p, k, n, m \in \mathbb{N}_0^d$, $|m| \leq j$, $|n| \leq j$ and (3.43) holds since this is exactly $\|\mathbf{T}\|^2_{L^2(\mathbf{U}_j,\mu_j)}$. For $\phi \in \tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$, we have

$$\begin{aligned}\langle T, \phi\rangle = \mathbf{T}(\mathfrak{J}_j(\phi)) = \int_{\mathbf{U}_j} \mathfrak{J}_j(\phi) \mathbf{T} d\mu_j \\ = \sum_{p,k\in\mathbb{N}_0^d} \int_{\mathbb{R}^d_+} F_{A,p,k}(x) x^{(p+k)/2} D^p \phi(x) dx + \sum_{|m|\leq j,\,|n|\leq j} \int_{\mathbb{R}^d_+} \tilde{F}_{A,n,m}(x) x^m D^n \phi(x) dx.\end{aligned}$$

The converse part follows trivially. □

Theorem 3.7.2. *Let $T \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$. Then, for each $A > 0$ there exist $j = j(A) \in \mathbb{N}_0$ and a set of $L^2(\mathbb{R}^d_+)$-functions*

$$\{F_{A,p,k}|\, p, k \in \mathbb{N}_0^d\} \cup \{\tilde{F}_{A,n,m}|\, n, m \in \mathbb{N}_0^d,\ |n| \leq j,\ |m| \leq j\} \tag{3.46}$$

such that (3.43) holds and the restriction of T to each $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ is given by (3.44).

If for each $A > 0$, there exist $j = j(A) \in \mathbb{N}_0$ and a set of $L^2(\mathbb{R}^d_+)$-functions (3.46) such that (3.43) holds, then for each $A > 0$ there exists $T_A \in (\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))'$ given by (3.44).

Furthermore, if for each $A_1 < A_2$, the restriction of T_{A_2} to $\tilde{G}^{\alpha,A_1}_{\alpha,A_1}(\mathbb{R}^d_+)$ coincides with T_{A_1}, then there exists $T \in (G^\alpha_\alpha(\mathbb{R}^d_+))'$ such that for each $A > 0$ the restriction of T to $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ is T_A, i.e. for $\phi \in \tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$, $\langle T, \phi\rangle$ is given by (3.44).

Proof. The first part follows directly from Proposition 3.7.1, since the restriction of T to each $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$, $A > 0$, is continuous.

For the second part, observe that the existence of $T_A \in (\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+))'$, for each $A > 0$, given by (3.44) is verified by Proposition 3.7.1.

Furthermore, if T_A, $A > 0$, satisfies that for each $A_1 < A_2$ the restriction of T_{A_2} to $\tilde{G}^{\alpha,A_1}_{\alpha,A_1}(\mathbb{R}^d_+)$ coincides with T_{A_1}, then one can define a linear functional

$$T : G^{\alpha}_{\alpha}(\mathbb{R}^d_+) \to \mathbb{C}, \quad \langle T, \phi \rangle = \langle T_A, \phi \rangle, \quad \phi \in \tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+).$$

Because of this condition, this is indeed a well defined linear mapping into $\mathbb{C}$. The continuity of T follows from the fact that each restriction of T to $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ is T_A, $A > 0$, which is continuous as a mapping from $\tilde{G}^{\alpha,A}_{\alpha,A}$ onto $\mathbb{C}$ and the fact that $G^{\alpha}_{\alpha}(\mathbb{R}^d_+)$ is the inductive limit of $\tilde{G}^{\alpha,A}_{\alpha,A}(\mathbb{R}^d_+)$ as $A \to \infty$. $\square$

Chapter 4

Weyl pseudo-differential operators with radial symbols

Concerning pseudo-differential operators, especially the Weyl calculus, we refer to the standard books [26] and [37].

This chapter is organized as follows.

Firstly, we will provide the motivations for introducing the Weyl pseudo-differential operator. We will give the formal derivation of formula for the Weyl pseudo-differential operator with symbol in $\mathcal{S}(\mathbb{R}^d)$.

Secondly, we will introduce the Wigner transform of functions in $\mathcal{S}(\mathbb{R}^d)$ as a tool to study the Weyl pseudo-differential operators with symbols in $\mathcal{S}'(\mathbb{R}^{2d})$.

Thirdly, we will refer to results of M. W. Wong [42] on the Weyl pseudo-differential operator on $L^2(\mathbb{R})$ with radial symbols by which we were motivated.

Finally, we will establish the continuity of the Weyl pseudo-differential operators with radial symbols, firstly, at the level of the symbol classes $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and $\mathcal{S}(\mathbb{R}^d_+)$, on the Gelfand-Shilov spaces and the Schwartz space. Then we consider the symbol classes $(G^\alpha_\alpha(\mathbb{R}^d_+))'$, $\alpha \geq 1$ and $\mathcal{S}'(\mathbb{R}^d_+)$ in order to extend the results on dual spaces of the Gelfand-Shilov spaces and dual space of the Schwartz space.

4.1 Problem of quantization

We present the main motivations for pseudo-differential operators from the point of view of quantum mechanics following [14, Section 14.3]. We refer to the pioneer work of H. Weyl [40, Chapter IV.14]. We explain the problem of quantization and we obtain the formula for the Weyl calculus of pseudo-differential operators.

In quantum mechanic, the observable quantities are represented by self-adjoint operators on a Hilbert space. In the standard model for a one-dimensional system the position variable q is represented by the multiplication operator $Xf(x) = xf(x)$, and the momentum variable p is represented bu the differentiation operator $Pf(x) = -if'(x)$. We can state the problem as follows: which operator should be associated to an arbitrary function $\sigma(q,p)$ on phase space. A quantization rule is a linear mapping $\sigma \to W_\sigma$ from functions $\sigma(q,p)$ on a phase space to operators on the given Hilbert space that extends the correspondence $q \to X$ and $p \to P$ to general functions on a phase space.

In his approach, Weyl considered the corresponding one-parameter subgroups of unitary operators $e^{iqX}f(x) = e^{iqx}f(x) = M_q f(x)$ and $e^{ixP}f(x) = f(x+p)$. Then he argued that the complex exponential $e^{i(x\cdot q+\xi\cdot p)}$ should correspond to the symmetric shift $M_{q/2}T_{-p}M_{q/2} = e^{-ipq/2}T_{-p}M_q$. Now, the Fourier inversion formula $f(x) = (2\pi)^{-d/2}\displaystyle\int_{\mathbb{R}^d} e^{ix\xi}\hat{f}(\xi)d\xi$, where $\hat{f}(\xi)$ is the Fourier transform of a function $f \in \mathcal{S}$, $\hat{f}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-ix\xi}\hat{f}(x)dx$ gives

$$\sigma(x,\xi) = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\sigma}(q,p)e^{i(x\cdot q+\xi\cdot p)}dpdq$$

and the linearity of the quantization procedure suggest that W_σ should be the operator

$$W_\sigma = (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\sigma}(q,p)e^{-iq\frac{p}{2}}T_{-p}M_q dpdq.$$

For us, it is more convenient to represent W_σ as an integral operator. Now, we give a formal derivation of the Weyl pseudo-differential operator with symbol $\sigma \in \mathcal{S}(\mathbb{R}^{2d})$

$$\begin{aligned}
(W_\sigma f)(x) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\sigma}(q,p)e^{-iq\frac{p}{2}}T_{-p}M_q f(x)dpdq \\
&= (2\pi)^{-2d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sigma(\omega,\xi)e^{-i(p\xi+q\omega)}e^{iqx+iq\frac{p}{2}}f(x+p)d\xi d\omega dpdq \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-ip\xi}\sigma(\omega,\xi)\delta(x+\frac{p}{2}-\omega)f(x+p)d\xi d\omega dp \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{-ip\xi}\sigma(x+\frac{p}{2},\xi)f(x+p)dpd\xi \\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}e^{i(x-y)\cdot\xi}\sigma\left(\frac{x+y}{2},\xi\right)f(y)dyd\xi, \quad f \in \mathcal{S}(\mathbb{R}^d). \qquad (4.1)
\end{aligned}$$

Remark 4.1.1. The Dirac delta function can be loosely thought of as a function on the real line which is zero everywhere except at the origin, where it is infinite,

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

which is also constrained to satisfy the identity

$$\int_{\mathbb{R}}\delta(x) = 1.$$

The delta function is an even distribution in the sense that $\delta(x) = \delta(-x)$. The delta function is said to "sift out" the value at $x = T$ i.e. $\displaystyle\int_{\mathbb{R}} f(x)\delta(x-T) = f(T)$. The inverse Fourier transform of the tempered distribution $f(\xi) = 1$ is the delta function. Formally, this is expressed as $\delta(x) = \displaystyle\int_{\mathbb{R}} e^{ix\xi}d\xi$.

4.2 Weyl pseudo-differential operators with symbols from $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))'$

Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then the function $W(f,g)$ defined on $\mathbb{R}^{2d}$ by

$$W(f,g)(x,\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi\cdot p} f\left(x+\frac{p}{2}\right) \overline{g\left(x-\frac{p}{2}\right)} dp, \quad x,\xi \in \mathbb{R}^d$$

is called the Wigner transform of f and g. The bilinear mapping $(f,g) \mapsto W(f,\overline{g})$, $\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^{2d})$ is continuous.

Corollary 4.2.1. *([42, Corollary 3.4.]) $W : \mathcal{S}(\mathbb{R}^n) \times \mathcal{S}\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{2n})$ can be extended uniquely to a bilinear operator*

$$W : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n})$$

such that

$$\|W(f,g)\|_{L^2(\mathbb{R}^{2n})} = \|f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)}$$

for all f and g from $L^2(\mathbb{R}^n)$.

Next theorem proves that the Gelfand-Shilov spaces are closed under the Wigner transform.

Theorem 4.2.1. *([38, Theorem 3.8, p. 179]) Let $f, g \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$, $\alpha \geq 1/2$. Then $W(f,g) \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$.*

Moreover, we have the following proposition.

Proposition 4.2.1. *A bilinear mapping*

$$(f,g) \mapsto W(f,\overline{g}),\ \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \times \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}),$$

is continuous.

Proof. Fix $g \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$. If we consider a mapping $f \mapsto W(f,\overline{g})$ as a mapping from $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^{2d})$ it is continuous since it decomposes as

$$\mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \longrightarrow \mathcal{S}(\mathbb{R}^d) \xrightarrow{f\mapsto W(f,\overline{g})} \mathcal{S}(\mathbb{R}^{2d}),$$

where the first mapping is the canonical inclusion. Hence, its graph is closed in $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^{2d})$. Since its image is in $\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$, its graph is closed in $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \times \mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$. As $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ is a (DFS)-space it is an ultrabornological and webbed space of De Wilde (see Proposition A.8.2). Now, the De Wilde closed graph theorem (see Theorem A.8.1) implies its continuity.

Similarly, for each fixed $f \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$, the mapping

$$g \mapsto W(f,\overline{g}),\ \ \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$$

is continuous.

Thus the bilinear mapping $(f,g) \mapsto W(f,\overline{g})$, $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \times \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$, is separately continuous and hence continuous since $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ is barrelled (DF)-space (see Theorem A.6.3). $\square$

The next theorem defines the notion of the Weyl pseudo-differential operator with a symbol in $\mathcal{S}'(\mathbb{R}^{2d})$.

Theorem 4.2.2. *([42, Theorem 12.1.]) For all $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ and $f \in \mathcal{S}(\mathbb{R}^d)$, $W_\sigma f \in \mathcal{S}'(\mathbb{R}^d)$.*

Let $\alpha \geq 1/2$. The Weyl pseudo-differential operator with a symbol $\sigma \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^{2d}))'$ defined by

$$(W_\sigma f)(g) = (2\pi)^{-d/2}\langle \sigma, W(f,\overline{g})\rangle \tag{4.2}$$

is a continuous and linear mapping from $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ into $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ (see [29, Theorem 2]).

4.3 Weyl pseudo-differential operator with radial symbols from $\mathcal{S}'(\mathbb{R}^2)$

For the Weyl pseudo-differential operator on $L^2(\mathbb{R})$ with radial symbols, a sufficient and necessary condition for boundedness is given in [42]. In order to obtain these conditions, we need the Wigner transform of Hermite functions on $\mathbb{R}$.

For $j,k = 0,1,2,...$, we define the function $\psi_{j,k}$ on $\mathbb{R}^2$ by

$$\psi_{j,k}(x,\xi) = W(h_j,h_k)(x,\xi), \quad x,\xi \in \mathbb{R}.$$

Theorem 4.3.1. *([42, Teorema 24.1.]) For $j,k = 0,1,2,...$ we get for any $\zeta = x+i\xi$,*

(i) $\psi_{j+k,j}(\zeta) = 2(-1)^j(2\pi)^{-\frac{1}{2}}\left(\frac{j!}{(j+k)!}\right)^{\frac{1}{2}}(\sqrt{2})^k(\bar{\zeta})^k L_j^k(2|\zeta|^2)e^{-|\zeta|^2}$,

(ii) $\psi_{j,j+k}(\zeta) = 2(-1)^j(2\pi)^{-\frac{1}{2}}\left(\frac{j!}{(j+k)!}\right)^{\frac{1}{2}}(\sqrt{2})^k\zeta^k L_j^k(2|\zeta|^2)e^{-|\zeta|^2}$.

Let σ be tempered function on $\mathbb{R}^2$. Suppose that σ is radial i.e.

$$\sigma(x,\xi) = \sigma(r), \quad x,\xi \in \mathbb{R},$$

where $r = \sqrt{x^2+\xi^2}$. Now, by Theorem 4.3.1, for $j,k = 0,1,2,...$ i $j \geq k$,

$$\psi_{j,k}(\zeta) = 2(-1)^k(2\pi)^{-\frac{1}{2}}\left(\frac{k!}{j!}\right)^{\frac{1}{2}}(\sqrt{2})^{j-k}(\bar{\zeta})^{j-k}L_k^{j-k}(2|\zeta|^2)e^{-|\zeta|^2} \tag{4.3}$$

for all $\zeta = x+i\xi$ u $\mathbb{C}$. Now, for all f,g in $\mathcal{S}(\mathbb{R})$, we obtain

$$\begin{aligned}(W_\sigma f)(\bar{g}) &= (2\pi)^{-1/2}\sigma(W(f,g))\\ &= (2\pi)^{-1/2}\sigma\Big(W\Big(f,\sum_{k=0}^{\infty}\langle g,h_k\rangle h_k\Big)\Big)\\ &= (2\pi)^{-1/2}\sigma\Big(\sum_{k=0}^{\infty}\langle \bar{g},h_k\rangle W(f,h_k)\Big)\\ &= (2\pi)^{-1/2}\sum_{k=0}^{\infty}\langle \bar{g},h_k\rangle \sigma(W(f,h_k))\\ &= (2\pi)^{-1/2}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\langle \bar{g},h_k\rangle\langle f,h_j\rangle \sigma(\psi_{j,k}). \qquad (4.4)\end{aligned}$$

Remark 4.3.1. Note that (4.4) is valid in the sense that we sum with respect to j first and then with respect to k.

Now, for $j,k=0,1,2,...$ i $j\geq k$, we obtain by (4.3),

$$\begin{aligned}\sigma(\psi_{j,k}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma(x,\xi)\psi_{j,k}(x,\xi)\,dx\,d\xi\\ &= \int_0^{2\pi}\int_0^{\infty}\sigma(\rho)2(-1)^k(2\pi)^{-\frac{1}{2}}\Big(\frac{k!}{j!}\Big)^{\frac{1}{2}}(\sqrt{2})^{j-k}(\rho)^{j-k}e^{-i(j-k)\theta}\\ &\quad\times L_k^{j-k}(2\rho^2)e^{-\rho^2}\rho\,d\rho\,d\theta\\ &= \int_0^{2\pi}e^{-i(j-k)\theta}\,d\theta\int_0^{\infty}\sigma(\rho)\Big(\frac{k!}{j!}\Big)^{\frac{1}{2}}2^{\frac{1}{2}(j-k)+1}(-1)^k(2\pi)^{-\frac{1}{2}}\\ &\quad\times L_k^{j-k}(2\rho^2)e^{-\rho^2}\rho^{j-k+1}\,d\rho.\end{aligned}$$

Hence,

$$\sigma(\psi_{j,k})=0,\quad j\neq k.$$

So,

$$(W_\sigma f)(\bar{g}) = (2\pi)^{-1/2}\sum_{k=0}^{\infty}\langle \bar{g},h_k\rangle\langle f,h_k\rangle\sigma(\psi_{k,k}), \qquad (4.5)$$

where

$$\sigma(\psi_{k,k}) = (2\pi)^{\frac{1}{2}}(-1)^k 2\int_0^{\infty}\sigma(\rho)L_k^0(2\rho^2)e^{-\rho^2}\rho\,d\rho,\quad k=0,1,2,...,$$

for all f,g in $\mathcal{S}(\mathbb{R})$.

Remark 4.3.2. The convergence in (4.5) is valid in the sense that the sequence of partial sums of the series is convergent.

Theorem 4.3.2. *([42, Theorem 24.5.,p.115]) Let σ be tempered function on $\mathbb{R}^2$. Suppose that σ is radial i.e.*

$$\sigma(x,\xi)=\sigma(r),\quad x,\xi\in\mathbb{R},$$

where $r = \sqrt{x^2+\xi^2}$. *For* $k = 0, 1, \dots$ *let*

$$a_k = \int_0^\infty \sigma(\rho) L_k^0(2\rho^2) e^{-\rho^2} \rho \, d\rho.$$

Then W_σ *is a bounded linear operator from* $L^2(\mathbb{R}^d)$ *into* $L^2(\mathbb{R}^d)$ *if and only if the sequence* $\{a_k\}_{k=0}^\infty$ *is bounded.*

Our goal is to prove Theorem 4.3.2 on $\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$, $\alpha \geq 1$ and their dual spaces, as well as on $\mathcal{S}(\mathbb{R}^d)$ and its dual space. As we shall see, we will obtain the assertions without assumptions on the sequence $\{a_k\}_{k=0}^\infty$ using the results obtained in Chapter 2 and Chapter 3.

4.4 Weyl pseudo-differential operators with radial symbols from G-type spaces and their dual spaces

In this section, we prove the continuity of the Weyl pseudo-differential operators with radial symbols from the spaces $G_\alpha^\alpha(\mathbb{R}_+^d)$, $\alpha \geq 1$ and their dual spaces on the Gelfand-Shilov spaces and their dula spaces.

Throughout the rest of this section, we denote by v the mapping $\mathbb{R}^{2d} \to \overline{\mathbb{R}_+^d}$, $(x,\xi) \mapsto v(x,\xi) = (x_1^2+\xi_1^2, \dots, x_d^2+\xi_d^2)$.

Proposition 4.4.1. *Let* $\sigma \in \mathcal{S}(\mathbb{R}_+^d)$. *Then* $\tilde{\sigma}(x,\xi) = \sigma \circ v(x,\xi) \in \mathcal{S}(\mathbb{R}^{2d})$.

Moreover, the mapping $\sigma \mapsto \tilde{\sigma} = \sigma \circ v$, $\mathcal{S}(\mathbb{R}_+^d) \to \mathcal{S}(\mathbb{R}^{2d})$, *is continuous.*

Proof. Fix $j \in \mathbb{N}$. For $p, q \in \mathbb{N}_0^d$, $|p| \leq j$ and $|q| \leq j$ observe that $D_x^p D_\xi^q \tilde{\sigma}(x,\xi)$ is a finite sum of the form $P(x,\xi) D_x^{p'} D_\xi^{q'} \sigma(v(x,\xi))$, where $P(x,\xi)$ are polynomials in (x,ξ) of degree at most $|p|+|q|$ which do not depend on σ (they only depend on the derivatives of v) and $p', q' \in \mathbb{N}_0^d$ are such that $p' \leq p$ and $q' \leq q$. Moreover, observe that the number of such terms that appear in $D_x^p D_\xi^q \tilde{\sigma}(x,\xi)$ depend only on p and q (and not on σ). For $p'', q'' \in \mathbb{N}_0^d$ we also have

$$\left| x^{p''} \xi^{q''} \right| \leq |x|^{|p''|} |\xi|^{|q''|} \leq (|x|^2 + |\xi|^2)^{(|p''|+|q''|)/2}.$$

Thus,

$$\sup_{\substack{|p''|\leq j \\ |q''|\leq j}} \sup_{\substack{|p|\leq j \\ |q|\leq j}} \sup_{(x,\xi)\in\mathbb{R}^{2d}} \left| x^{p''} \xi^{q''} D_x^p D_\xi^q \tilde{\sigma}(x,\xi) \right| \leq C \sup_{\substack{|n|\leq 2j \\ |m|\leq 2j}} \sup_{t\in\mathbb{R}_+^d} \left| t^m D^n \sigma(t) \right|.$$

Hence, $\tilde{\sigma} \in \mathcal{S}(\mathbb{R}^{2d})$ and the mapping

$$\sigma \mapsto \tilde{\sigma} = \sigma \circ v, \ \mathcal{S}(\mathbb{R}_+^d) \to \mathcal{S}(\mathbb{R}^{2d})$$

is continuous. □

Let $\alpha \geq 1/2$ and $\sigma(\rho) \in G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)$. Denote by $\sigma_0(\rho) = \sigma(2\rho)$, $\rho \in \mathbb{R}^d_+$. Then the functions $\tilde{\sigma}$ and $\tilde{\sigma}_0$ defined by

$$\tilde{\sigma}(x,\xi) = \sigma \circ v(x,\xi),\ \tilde{\sigma}_0(x,\xi) = \sigma_0 \circ v(x,\xi), \qquad (x,\xi) \in \mathbb{R}^{2d} \tag{4.6}$$

belong to $\mathcal{S}(\mathbb{R}^{2d})$ (see Proposition 4.4.1). Hence, the Weyl pseudo-differential operator with a symbol $\tilde{\sigma}_0$ is a continuous mapping from $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ onto $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$.

Theorem 4.4.1. *Let $\alpha \geq 1/2$ and $\sigma(\rho) \in G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)$. Denote by $\sigma_0(\rho) = \sigma(2\rho)$, $\rho \in \mathbb{R}^d_+$. Let $\tilde{\sigma}, \tilde{\sigma}_0 \in \mathcal{S}(\mathbb{R}^{2d})$ be the functions defined in (4.6). Then*

$$W_{\tilde{\sigma}_0} : \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$$

is a continuous mapping and it extends to a continuous mapping

$$W_{\tilde{\sigma}_0} : (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))' \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^d).$$

If $f, g \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$ and

$$f_k = \langle f, h_k\rangle, g_k = \langle g, h_k\rangle \text{ and } \sigma_k = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\int_{\mathbb{R}^d_+} \sigma(\rho)\mathcal{L}_k(\rho)d\rho,$$

then

$$(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{k \in \mathbb{N}^d_0} f_k g_k \sigma_k.$$

Moreover, if

$$\sigma_{0,j}(\eta) \xrightarrow{G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)} \sigma_0(\eta) \qquad \text{as } j \to \infty$$

then $W_{\tilde{\sigma}_{0,j}} \to W_{\tilde{\sigma}_0}$ in the strong topology of $\mathcal{L}((\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))', \mathcal{S}^\alpha_\alpha(\mathbb{R}^d))$.

Proof. First we $W_{\tilde{\sigma}_0}$ of $f, g \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$. Since $\sum_{n \in \mathbb{N}^d_0} f_n h_n$ and $\sum_{n \in \mathbb{N}^d_0} g_n h_n$ converge absolutely to f and g in $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$, respectively (cf. Proposition 1.6.1) and the mapping

$$(\varphi, \psi) \mapsto W(\varphi, \overline{\psi}),\ \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \times \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}),$$

is continuous (see Proposition 4.2.1), we conclude

$$W(f, \overline{g}) = \sum_{(m,k) \in \mathbb{N}^{2d}_0} f_m g_k W(h_m, h_k),$$

where the sum converges absolutely in $\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$. As $\tilde{\sigma}_0 \in \mathcal{S}(\mathbb{R}^{2d}) \subseteq (\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))'$, we have

$$(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2} \sum_{(m,k) \in \mathbb{N}^{2d}_0} f_m g_k \langle \tilde{\sigma}_0, \psi_{m,k}\rangle, \tag{4.7}$$

where $\psi_{m,k} = W(h_m, h_k)$. Clearly,

$$\psi_{m,k} = \prod_{r=1}^d \psi_{m_r,k_r}, \quad \text{where} \quad \psi_{m_r,k_r} = W(h_{m_r}, h_{k_r}).$$

Using Theorem 4.3.1 and denoting $\eta_r = x_r + i\xi_r \in \mathbb{C}$, we have

(i) If $m_r \geq k_r$

$$\psi_{m_r,k_r}(x_r,\xi_r) = 2(-1)^{k_r}(2\pi)^{-1/2}\left(\frac{k_r!}{m_r!}\right)^{1/2}(\sqrt{2})^{m_r-k_r}(\overline{\eta_r})^{m_r-k_r} \times L_{k_r}^{m_r-k_r}(2|\eta_r|^2)e^{-|\eta_r|^2}. \tag{4.8}$$

(ii) If $k_r \geq m_r$

$$\psi_{m_r,k_r}(x_r,\xi_r) = 2(-1)^{m_r}(2\pi)^{-1/2}\left(\frac{m_r!}{k_r!}\right)^{1/2}(\sqrt{2})^{k_r-m_r}\eta_r^{k_r-m_r} \times L_{m_r}^{k_r-m_r}(2|\eta_r|^2)e^{-|\eta_r|^2}. \tag{4.9}$$

In terms of polar coordinates the integral

$$\langle\tilde{\sigma}_0,\psi_{m,k}\rangle = \int_{\mathbb{R}^{2d}}\sigma_0(v(x,\xi))\psi_{m,k}(x,\xi)dxd\xi$$

is

$$\langle\tilde{\sigma}_0,\psi_{m,k}\rangle = C_{m,k}\prod_{r=1}^{d}\int_{-\pi}^{\pi}e^{-i(m_r-k_r)\theta_r}d\theta_r.$$

Thus $\langle\tilde{\sigma}_0,\psi_{m,k}\rangle = 0$ when $m \neq k$. Moreover,

$$\begin{aligned}\langle\tilde{\sigma}_0,\psi_{k,k}\rangle &= (2\pi)^{d/2}(-1)^{|k|}2^d\int_{\mathbb{R}^d_+}\sigma(2\rho_1^2,\ldots,2\rho_d^2)L_k(2\rho_1^2,\ldots,2\rho_d^2)e^{-|\rho|^2}\rho^{\mathbf{1}}d\rho\\ &= (2\pi)^{d/2}(-1)^{|k|}2^{-d}\int_{\mathbb{R}^d_+}\sigma(y)\mathcal{L}_k(y)dy = \sigma_k.\end{aligned}$$

By (4.7), we obtain

$$(W_{\tilde{\sigma}_0}f)(g) = (2\pi)^{-d/2}\sum_{k\in\mathbb{N}_0^d}f_kg_k\sigma_k \tag{4.10}$$

and the series converges absolutely since $\{f_n\}_{n\in\mathbb{N}_0^d}, \{g_n\}_{n\in\mathbb{N}_0^d}, \{\sigma_n\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$ (since $f,g \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$, $\sigma \in G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)$).

Let now $f,g \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$. Define

$$(W_{\tilde{\sigma}_0}f)(g) = (2\pi)^{-d/2}\sum_{n\in\mathbb{N}_0^d}f_ng_n\sigma_n.$$

Observe that the series converges absolutely since $\{f_n\}_{n\in\mathbb{N}_0^d}, \{g_n\}_{n\in\mathbb{N}_0^d} \in (s^{2\alpha})'$ and $\{\sigma_n\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$ ($\sigma \in G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)$; cf. Theorem 3.4.1). Thus, if we fix $f \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$, the mapping

$$g \mapsto (W_{\tilde{\sigma}_0}f)(g),\ (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))' \to \mathbb{C},$$

is a well defined linear mapping. To prove that it is continuous let B be a bounded subset of $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$. Thus for each $a > 1$ there exists $C > 0$ such that

$$|g_k| \leq Ca^{|k|^{1/(2\alpha)}},\ \forall k \in \mathbb{N}_0^d,\ \forall g \in B.$$

Hence,

$$\sup_{g\in B} |(W_{\tilde{\sigma}_0} f)(g)| < \infty,$$

i.e. $W_{\tilde{\sigma}_0} f$ maps bounded subsets in $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ into bounded subsets of $\mathbb{C}$. Since $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ is bornological,

$$g \mapsto (W_{\tilde{\sigma}_0} f)(g)\ ,(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \to \mathbb{C},$$

is continuous. Hence $W_{\tilde{\sigma}_0} f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ ($\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ is reflexive). Now we conclude that

$$W_{\tilde{\sigma}_0} f = \sum_{n\in\mathbb{N}_0^d} f_n \sigma_n h_n.$$

This is exactly Hermite expansion of $W_{\tilde{\sigma}_0} f$; $\{f_n\sigma_n\}_n \in s^{2\alpha}$. Thus, the mapping

$$f \mapsto W_{\tilde{\sigma}_0} f,\ (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \to \mathcal{S}_\alpha^\alpha(\mathbb{R}^d),$$

is well defined and linear. Arguing similarly as before, one can prove that when f varies in a bounded subset B of $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$, the set

$$\{\{f_k\sigma_k\}_{k\in\mathbb{N}_0^d} |\, f \in B\} \text{ is bounded in } s^{2\alpha}.$$

Thus,

$$\{W_{\tilde{\sigma}_0} f |\, f \in B\} \text{ is bounded in } \mathcal{S}_\alpha^\alpha(\mathbb{R}^d).$$

As $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ is bornological, the mapping

$$f \mapsto W_{\tilde{\sigma}_0} f,\ (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))' \to \mathcal{S}_\alpha^\alpha(\mathbb{R}^d),$$

is continuous. Observe that $W_{\tilde{\sigma}_0} f$ coincides with the Weyl transform of f when $f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (cf. (4.10)).

If $\sigma_j \to \sigma$ as $j \to \infty$, in $G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d)$, Theorem 3.4.1 implies that

$$\{\sigma_{n,j}\}_{n\in\mathbb{N}_0^d} \xrightarrow{s^{2\alpha}} \{\sigma_n\}_{n\in\mathbb{N}^d},\ j \to \infty$$

and since the latter is a (DFN)-space, the convergence also holds in $s^{2\alpha,a}$ for some $a > 1$. Thus, for each fixed $f \in (\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$,

$$\{f_n\sigma_{n,j}\}_n \xrightarrow{s^{2\alpha}} \{f_n\sigma_n\}_n.$$

Hence,

$$\sum_{n\in\mathbb{N}_0^d} f_n\sigma_{n,j} h_n \xrightarrow{\mathcal{S}_\alpha^\alpha(\mathbb{R}^d)} \sum_{n\in\mathbb{N}_0^d} f_n\sigma_n h_n.$$

Since we have obtained that $W_{\tilde{\sigma}_{0,j}} \to W_{\tilde{\sigma}_0}$ in the topology of simple convergence in $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$, by the Banach-Steinhaus theorem it follows that the convergence holds in the topology of precompact convergence. As $(\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))'$ is a Montel space, the convergence also holds in the strong topology of $\mathcal{L}((\mathcal{S}_\alpha^\alpha(\mathbb{R}^d))', \mathcal{S}_\alpha^\alpha(\mathbb{R}^d))$. $\square$

Let $\alpha \geq 1/2$. If σ is a measurable function on $\mathbb{R}^d_+$ such that

$$\frac{\sigma(\rho)}{(\mathbf{1}+\rho)^{n/2}} \in L^2(\mathbb{R}^d_+)$$

for some $n \in \mathbb{N}^d_0$ then one easily verifies that $\sigma \in (\mathcal{S}(\mathbb{R}^d_+))'$. Since the canonical inclusion $G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+) \to \mathcal{S}(\mathbb{R}^d_+)$ is continuous and dense, $(\mathcal{S}(\mathbb{R}^d_+))'$ is continuously injected into $(G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$, hence $\sigma \in (G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$.

Lemma 4.4.1. *Let $\alpha \geq 1/2$ and σ_n, $n \in \mathbb{N}^d_0$, be measurable functions on $\mathbb{R}^d_+$ such that $\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2} \in L^2(\mathbb{R}^d_+)$, for all $n \in \mathbb{N}^d_0$ and*

$$\sum_{n\in\mathbb{N}^d_0} \left\|\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2}\right\|_{L^2(\mathbb{R}^d_+)} A^{|n|} n^{\alpha n} < \infty, \quad \textit{for each } A > 0.$$

Then $\sum_{n\in\mathbb{N}^d_0} \sigma_n$ converges absolutely in $(G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$.

Furthemore, $\tilde{\sigma}_n(x,\xi) = \sigma_n(2v(x,\xi))$, for all $n \in \mathbb{N}^d_0$, is measurable on $\mathbb{R}^{2d}$ and

$$\frac{\tilde{\sigma}_n(x,\xi)}{(\mathbf{1}+2v(x,\xi))^{n/2}} \in L^2(\mathbb{R}^{2d}).$$

Moreover, $\sum_{n\in\mathbb{N}^d_0} \tilde{\sigma}_n(x,\xi)$ converges absolutely in $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))'$.

Proof. Firstly, we will prove that $\sum_{n\in\mathbb{N}^d_0} \sigma_n$ converges absolutely in $(G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$. Let B be bounded subset of $G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+)$. For each $f \in B$ denote by $a_{n,f} = \langle f, \mathcal{L}_n\rangle$. By Theorem 3.4.1, $\{\{a_{n,f}\}_{n\in\mathbb{N}^d_0} | \, f \in B\}$ is bounded in $s^{2\alpha}$ and hence also bounded in $s^{2\alpha,a}$ for some $a > 1$, i.e. there exists $C_0 > 0$ such that

$$|a_{n,f}| \leq C_0 a^{-|n|^{1/(2\alpha)}}, \quad \text{for all } f \in B.$$

For $f \in B$, $n \in \mathbb{N}^d_0$, we have

$$\begin{aligned}
|\langle \sigma_n, f\rangle| &\leq \sum_{k\in\mathbb{N}^d_0} |a_{k,f}| \int_{\mathbb{R}^d_+} |\sigma_n(\rho)||\mathcal{L}_k(\rho)| d\rho \\
&\leq C_0 \left\|\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2}\right\|_{L^2(\mathbb{R}^d_+)} \sum_{k\in\mathbb{N}^d_0} a^{-|k|^{1/(2\alpha)}} \sum_{m\leq n} \binom{n}{m} \|\rho^{m/2}\mathcal{L}_k\|_{L^2(\mathbb{R}^d_+)}.
\end{aligned}$$

As in the first part of the proof of Proposition 3.3.1, by (3.11), there exist $C_1, A > 1$ which depend on a but not on $n \in \mathbb{N}^d_0$ such that

$$\sum_{k\in\mathbb{N}^d_0} a^{-|k|^{1/(2\alpha)}} \sum_{m\leq n} \binom{n}{m} \|\rho^{m/2}\mathcal{L}_k\|_{L^2(\mathbb{R}^d_+)} \leq C_1 A^{|n|} n^{\alpha n}.$$

Hence, by the assumption on σ_n, $n \in \mathbb{N}^d_0$, we have

$$\sum_{n\in\mathbb{N}^d_0} \sup_{f\in B} |\langle \sigma_n, f\rangle| < \infty,$$

i.e. $\sum_{n\in\mathbb{N}_0^d}\sigma(\rho)$ converges absolutely in $(G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$.

Next we will prove that for each $n\in\mathbb{N}_0^d$, $\tilde{\sigma}_n$ is measurable on $\mathbb{R}^{2d}$. Firstly, we will show the following:

Let $v_1:\mathbb{R}^{2d}\to\overline{\mathbb{R}_+^d}$ be defined by $v_1(x,\xi)=(2x_1^2+2\xi_1^2,\dots,2x_d^2+2\xi_d^2)$. If $g:\overline{\mathbb{R}_+^d}\to\mathbb{C}$ is measurable then $f:\mathbb{R}^{2d}\to\mathbb{C}$, $f=g\circ v_1$, is also measurable.

For brevity in notation we denote by λ_d and λ_{2d} the Lebesgue measure on $\mathbb{R}^d$ and $\mathbb{R}^{2d}$, respectively. We will prove that if $N\subseteq\overline{\mathbb{R}_+^d}$ with $\lambda_d(N)=0$ then $\lambda_{2d}(v_1^{-1}(N))=0$. Observe that this implies the measurability of f since:

- Every measurable set is the union of a Borel set and a set of measure zero and
- the preimage of every Borel set under v_1 is Borel set (since v_1 is continuous).

Let $N\subseteq\overline{\mathbb{R}_+^d}$, with $\lambda_d(N)=0$. Denote by $N_1=N\cap\mathbb{R}_+^d$ and by $N_2=N\backslash N_1$. Obviously,

$$\lambda_{2d}\left(v_1^{-1}(\overline{\mathbb{R}_+^d}\backslash\mathbb{R}_+^d)\right)=0.$$

Thus $v_1^{-1}(N_2)$ is measurable and has measure zero.

It remains to prove that $\lambda_{2d}(v_1^{-1}(N_1))=0$. Let $\varepsilon>0$ be arbitrary but fixed. Since $\lambda_d(N_1)=0$, there exists an open set $O\subseteq\mathbb{R}_+^d$, such that $N_1\subseteq O$ and $\lambda_d(O)<\varepsilon/\pi^d$. There exist countable number of cubes

$$B(\rho^{(j)},r_j)=\{\rho\in\mathbb{R}_+^d|\,\rho_l^{(j)}\leq\rho_l<\rho_l^{(j)}+r_j,\ l=1,\dots,d\},\ j\in\mathbb{N}$$

which are pairwise disjoint and

$$O=\bigcup_{j\in\mathbb{N}}B(\rho^{(j)},r_j)$$

(see [35, p. 49]). Observe that

$$\varepsilon/\pi^d>\lambda_d(O)=\sum_{j\in\mathbb{N}}\lambda_d(B(\rho^{(j)},r_j))=\sum_{j\in\mathbb{N}}r_j^d$$

and

$$v_1^{-1}(B(\rho^{(j)},r_j))=\prod_{l=1}^{d}\left\{(x_l,\xi_l)|\,\rho_l^{(j)}/2\leq x_l^2+\xi_l^2<\rho_l^{(j)}/2+r_j/2\right\}.$$

Thus $\lambda_{2d}(v_1^{-1}(B(\rho^{(j)},r_j))=r_j^d\pi^d/2^d$. Hence,

$$\lambda_{2d}(v_1^{-1}(O))=\sum_{j\in\mathbb{N}}r_j^d\pi^d/2^d<\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that $v_1^{-1}(N_1)$ is measurable and it has measure zero. Hence, the measurability of $\tilde{\sigma}_n$ follows.

Moreover,

$$\left\|\tilde{\sigma}_n(x,\xi)/(\mathbf{1}+2v(x,\xi))^{n/2}\right\|_{L^2(\mathbb{R}^{2d})}^2=2^{-d}\pi^d\left\|\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2}\right\|_{L^2(\mathbb{R}_+^d)}^2.$$

Clearly, $\tilde{\sigma}_n \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))'$ for each $n \in \mathbb{N}^d_0$.

To prove that $\sum_{n\in\mathbb{N}^d_0} \tilde{\sigma}_n$ converges absolutely in $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))'$, let B be a bounded subset of $\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d})$. As the latter space is the inductive limit of $\varinjlim_{A\to\infty} \mathcal{S}^{\alpha,A}_{\alpha,A}(\mathbb{R}^{2d})$ with compact linking mappings, there exist $C, A \geq 1$ such that for all $f \in B$

$$\left\|x^n \xi^m D^p_x D^q_\xi f(x,\xi)\right\|_{L^2(\mathbb{R}^{2d})} \leq CA^{|n+m+p+q|} n!^\alpha m!^\alpha p!^\alpha q!^\alpha, \quad \forall n,m,p,q \in \mathbb{N}^d_0.$$

Next, for $f \in B$, we have

$$\begin{aligned}
|\langle \tilde{\sigma}_n, f\rangle| &\leq \left\|\tilde{\sigma}_n(x,\xi)/(\mathbf{1}+2v(x,\xi))^{n/2}\right\|_{L^2(\mathbb{R}^{2d})} \left\|f(x,\xi)(\mathbf{1}+2v(x,\xi))^{n/2}\right\|_{L^2(\mathbb{R}^{2d})} \\
&\leq \pi^d 2^{|n|} \left\|\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2}\right\|^2_{L^2(\mathbb{R}^d_+)} \sum_{m+k+p=n} \frac{n!}{m!k!p!} \left\|x^m \xi^k f(x,\xi)\right\|_{L^2(\mathbb{R}^{2d})} \\
&\leq C\pi^d (6A)^{|n|} n!^\alpha \left\|\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2}\right\|^2_{L^2(\mathbb{R}^d_+)}.
\end{aligned}$$

Hence, by the assumption in the lemma,

$$\sum_{n\in\mathbb{N}^d_0} \sup_{f\in B} |\langle \tilde{\sigma}_n, f\rangle| < \infty,$$

i.e. $\sum_{n\in\mathbb{N}^d_0} \tilde{\sigma}_n$ absolutely converges in $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))'$. $\square$

Let σ_n and $\tilde{\sigma}_n$, $n \in \mathbb{N}^d_0$, be as in the previous lemma and

$$\tilde{\sigma}(x,\xi) = \sum_{n\in\mathbb{N}^d_0} \tilde{\sigma}_n(x,\xi) \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'.$$

The Weyl pseudo-differential operator $W_{\tilde{\sigma}}$ is a continuous mapping from $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ into $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$. In this case, we obtain improvement with the following result.

Theorem 4.4.2. *Let $\alpha \geq 1/2$. Let $\sigma_n(\rho)$ and $\tilde{\sigma}_n(x,\xi) = \sigma_n(2v(x,\xi))$, $n \in \mathbb{N}^d_0$, be as in Lemma 4.4.1. Then*

$$W_{\tilde{\sigma}} : \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^d),$$

where

$$\tilde{\sigma}(x,\xi) = \sum_{n\in\mathbb{N}^d_0} \tilde{\sigma}_n(x,\xi) \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^{2d}))',$$

is a continuous mapping and it extends to a continuous mapping

$$W_{\tilde{\sigma}} : (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))' \to (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'.$$

Next, assume that for each $j \in \mathbb{N}$, $\sigma^{(j)}_n \in (G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$, $n \in \mathbb{N}^d_0$, be as in Lemma 4.4.1. Denote by $\sigma^{(j)} = \sum_{n\in\mathbb{N}^d_0} \sigma^{(j)}_n \in (G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$. If

$$\sigma^{(j)}(\eta) \xrightarrow{(G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'} \sigma(\eta), \qquad \text{as } j \to \infty,$$

with σ as above, then $W_{\tilde{\sigma}^{(j)}} \to W_{\tilde{\sigma}}$ in the strong topology of $\mathcal{L}(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d), \mathcal{S}^\alpha_\alpha(\mathbb{R}^d))$ and $\mathcal{L}((\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))', (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))')$.

Proof. Denote by $\sigma = \sum_{n\in\mathbb{N}_0^d} \sigma_n \in (G_{2\alpha}^{2\alpha}(\mathbb{R}_+^d))'$ (cf. Lemma 4.4.1). Let $f, g \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ and denote by

$$f_k = \langle f, h_k\rangle,\ g_k = \langle g, h_k\rangle \text{ and } s_k = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle \sigma, \mathcal{L}_k\rangle.$$

Similarly as in the first part of the proof of Theorem 4.4.1, one obtains

$$(W_{\tilde{\sigma}} f)(g) = (2\pi)^{-d/2} \sum_{(m,k)\in\mathbb{N}_0^{2d}} f_m g_k \langle \tilde{\sigma}, \psi_{m,k}\rangle,$$

where $\psi_{m,k} = W(h_m, h_k)$ and the sum converges absolutely. Next,

$$\langle \tilde{\sigma}(x,\xi), \psi_{m,k}(x,\xi)\rangle = \sum_{n\in\mathbb{N}_0^d} \int_{\mathbb{R}^{2d}} \sigma_n(2v(x,\xi))\psi_{m,k}(x,\xi)dxd\xi.$$

By the same technique as in the proof of Theorem 4.4.1,

$$\int_{\mathbb{R}^{2d}} \sigma_n(2v(x,\xi))\psi_{m,k}(x,\xi)dxd\xi = C_{n,m,k}\prod_{r=1}^{d}\int_{-\pi}^{\pi} e^{-i(m_r-k_r)\theta_r}d\theta_r.$$

Thus $\langle \tilde{\sigma}, \psi_{m,k}\rangle = 0$ for $m \neq k$. Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \sigma_n(2v(x,\xi))\psi_{k,k}(x,\xi)dxd\xi \\
&= (2\pi)^{d/2}(-1)^{|k|}2^d \int_{\mathbb{R}_+^d} \sigma_n(2\rho_1^2 \dots, 2\rho_d^2)L_k(2\rho_1^2, \dots, 2\rho_d^2)e^{-|\rho|^2}\rho^{\mathbf{1}}d\rho \\
&= (2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle \sigma_n, \mathcal{L}_k\rangle.
\end{aligned}$$

Thus,

$$\langle \tilde{\sigma}(x,\xi), \psi_{k,k}(x,\xi)\rangle = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle \sigma, \mathcal{L}_k\rangle = s_k.$$

Hence, we obtain

$$(W_{\tilde{\sigma}} f)(g) = (2\pi)^{-d/2}\sum_{k\in\mathbb{N}_0^d} f_k g_k s_k$$

and the series converges absolutely since $\{f_k\}_{k\in\mathbb{N}_0^d}, \{g_k\}_{k\in\mathbb{N}_0^d} \in s^{2\alpha}$ (see Proposition 1.6.1) and $\{s_k\}_{k\in\mathbb{N}_0^d} \in (s^{2\alpha})'$ (see Theorem 3.4.2). Observe that for each $n \in \mathbb{N}_0^d$, $(W_{\tilde{\sigma}} f)(h_n) = f_n s_n$. Since $\{s_n\}_{n\in\mathbb{N}_0^d} \in (s^{2\alpha})'$ and $\{f_n\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$, we have $\{f_n s_n\}_{n\in\mathbb{N}_0^d} \in s^{2\alpha}$, i.e. $W_{\tilde{\sigma}} f \in \mathcal{S}_\alpha^\alpha(\mathbb{R}^d)$ (by Proposition 1.6.1). We conclude that

$$f \mapsto W_{\tilde{\sigma}} f,\ \mathcal{S}_\alpha^\alpha(\mathbb{R}^d) \to \mathcal{S}_\alpha^\alpha(\mathbb{R}^d),$$

is a well defined linear mapping. Moreover,

$$W_{\tilde{\sigma}} f = \sum_{n\in\mathbb{N}_0^d} f_n s_n h_n.$$

To prove the continuity let B be a bounded subset of $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$. As $\{s_k\}_{k\in\mathbb{N}^d_0} \in (s^{2\alpha})'$, the set

$$\{\{f_n s_n\}_{n\in\mathbb{N}^d_0} | \, f \in B\} \text{ is bounded in } s^{2\alpha},$$

thus

$$\{W_{\tilde{\sigma}} f | \, f \in B\} \text{ is bounded in } \mathcal{S}^\alpha_\alpha(\mathbb{R}^d).$$

As $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ is bornological,

$$f \mapsto W_{\tilde{\sigma}} f, \;\; \mathcal{S}^\alpha_\alpha(\mathbb{R}^d) \to \mathcal{S}^\alpha_\alpha(\mathbb{R}^d),$$

is continuous. By similar technique, one proves that

$$W_{\tilde{\sigma}} f = \sum_{n\in\mathbb{N}^d_0} f_n s_n h_n \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))', \text{ for each } f \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$$

and the mapping,

$$f \mapsto W_{\tilde{\sigma}} f, \; (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))' \to (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))',$$

is continuous.

Let $\sigma, \sigma^{(j)} \in (G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$, $j \in \mathbb{N}$, be as assumed in the theorem, with $\sigma^{(j)} \to \sigma$ in $(G^{2\alpha}_{2\alpha}(\mathbb{R}^d_+))'$. In order to prove $W_{\tilde{\sigma}^{(j)}} \to W_{\tilde{\sigma}}$ in the strong topology of $\mathcal{L}(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d), \mathcal{S}^\alpha_\alpha(\mathbb{R}^d))$ (resp. in the strong topology of $\mathcal{L}((\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))', (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'))$ it is enough to prove that for each $f \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ (resp. for each $f \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$), $W_{\tilde{\sigma}^{(j)}} f \to W_{\tilde{\sigma}} f$ in $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ (resp. in $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$) since in this case the Banach-Steinhaus theorem implies convergence in the topology of precompact convergence. As $\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ (resp. $(\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$) is Montel the convergence also holds in the strong topology. Thus for the fixed $f \in \mathcal{S}^\alpha_\alpha(\mathbb{R}^d)$ (resp. $f \in (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'$). Theorem 3.4.2 implies that

$$\{s^{(j)}_k\}_{k\in\mathbb{N}^d_0} \xrightarrow{(s^{2\alpha})'} \{s_k\}_{k\in\mathbb{N}^d_0}.$$

Then

$$\{f_k s^{(j)}_k\}_{k\in\mathbb{N}^d_0} \xrightarrow{s^{2\alpha}} \{f_k s_k\}_{k\in\mathbb{N}^d_0}, \text{ (resp. in } (s^{2\alpha})')$$

i.e.

$$W_{\tilde{\sigma}^{(j)}} f \xrightarrow{\mathcal{S}^\alpha_\alpha(\mathbb{R}^d)} W_{\tilde{\sigma}} f, \text{ (resp. in } (\mathcal{S}^\alpha_\alpha(\mathbb{R}^d))'.$$

□

Remark 4.4.1. Let σ_n, $n \in \mathbb{N}^d_0$, be measurable functions on $\mathbb{R}^d_+$ such that $\sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2} \in L^2(\mathbb{R}^d_+)$, for all $n \in \mathbb{N}^d_0$ and for each $A > 0$,

$$\sum_{n\in\mathbb{N}^d_0} \left\| \sigma_n(\rho)/(\mathbf{1}+\rho)^{n/2} \right\|_{L^2(\mathbb{R}^d_+)} A^{|n|} n^{\alpha n/2} < \infty.$$

Then, by Lemma 4.4.1 $\sum_{n\in\mathbb{N}^d_0} \sigma_n$ converges absolutely in $(G^\alpha_\alpha(\mathbb{R}^d_+))'$ to some σ. Moreover, the same result also states that

$$\tilde{\sigma}_n(x,\xi) = \sigma_n(2x_1^2 + 2\xi_1^2, \dots, 2x_d^2 + 2\xi_d^2)$$

is measurable on $\mathbb{R}^{2d}$ and $\sum_n \tilde{\sigma}_n$ converges absolutely in $(\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^{2d}))'$ to some $\tilde{\sigma}$. The Weyl pseudodifferential operator with symbol $\tilde{\sigma}$ is well defined and continues mapping from $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d)$ into $\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d)$, it extends to a continuous mapping from $(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'$ to $(\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))'$. It is given by

$$W_{\tilde{\sigma}} f = \sum_k f_k \sigma_k h_k, \quad f = \sum_k f_k h_k \in (\mathcal{S}^{\alpha}_{\alpha}(\mathbb{R}^d))', \; \sigma_k = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle \sigma, l_k\rangle$$

and $\sigma_k = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\langle \sigma, l_k\rangle$ (see Theorem 4.4.2). By Theorem 3.6.2 each σ given as above originates from a unique even tempered ultradistribution by the isomorphism $({}^t\mathfrak{I})^{-1} : (\mathcal{S}^{\alpha/2}_{\alpha/2,\,\mathrm{even}}(\mathbb{R}^d))' \to (G^{\alpha}_{\alpha}(\mathbb{R}^d_+))'$.

4.5 Weyl pseudo-differential operator with radial symbols from $\mathcal{S}(\mathbb{R}^d_+)$ and its dual space

In this section, we prove, by the similar arguments as in Section 4.4, the continuity of the Weyl pseudo-differential operators with radial symbols from $\mathcal{S}(\mathbb{R}^d_+)$ and its dual space on the Schwartz space and its dual space.

Theorem 4.5.1. *Let $\sigma(\rho) \in \mathcal{S}(\mathbb{R}^d_+)$ and denote by $\sigma_0(\rho) = \sigma(2\rho)$, $\rho \in \mathbb{R}^d_+$. Let $\tilde{\sigma}, \tilde{\sigma}_0 \in \mathcal{S}(\mathbb{R}^{2d})$ be the functions defined in (4.6). Then*

$$W_{\tilde{\sigma}_0} : (\mathcal{S}(\mathbb{R}^d))' \to \mathcal{S}(\mathbb{R}^d)$$

extends to a continuous mapping. If $f, g \in (\mathcal{S}(\mathbb{R}^d))'$ and

$$f_k = \langle f, h_k\rangle, g_k = \langle g, h_k\rangle \text{ and } \sigma_k = (2\pi)^{d/2}(-1)^{|k|}2^{-d}\int_{\mathbb{R}^d_+} \sigma(\rho)\mathcal{L}_k(\rho)d\rho,$$

then

$$(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-d/2}\sum_{k\in\mathbb{N}^d_0} f_k g_k \sigma_k.$$

Moreover, if

$$\sigma_{0,j}(\eta)x \to \mathcal{S}(\mathbb{R}^d_+)\sigma_0(\eta) \qquad \text{as } j \to \infty$$

then $W_{\tilde{\sigma}_{0,j}} \to W_{\tilde{\sigma}_0}$ in the strong topology of $\mathcal{L}((\mathcal{S}(\mathbb{R}^d))', \mathcal{S}(\mathbb{R}^d))$.

Proof. First we compute the Weyl transform $W_{\tilde{\sigma}_0}$ of $f \in \mathcal{S}(\mathbb{R}^d)$. Let $g \in \mathcal{S}(\mathbb{R}^d)$. Following (4.4) we obtain

$$\begin{aligned}(W_{\tilde{\sigma}_0} f)(g) &= (2\pi)^{-d/2}\tilde{\sigma}_0(W(f,\overline{g})) \\ &= (2\pi)^{-d/2}\sum_{k\in\mathbb{N}^d_0}\sum_{j\in\mathbb{N}^d_0}\langle g, h_k\rangle\langle f, h_j\rangle\tilde{\sigma}_0(\psi_{j,k}). \end{aligned} \tag{4.11}$$

Using (4.8) and (4.9) and passing to polar coordinate in the integral

$$\tilde{\sigma}_0(\psi_{j,k}) = \int_{\mathbb{R}^{2d}} \sigma_0(v(x,\xi))\psi_{j,k}(x,\xi)dxd\xi$$

one easily obtains that

$$\tilde{\sigma}_0(\psi_{j,k}) = C_{j,k} \prod_{r=1}^{d} \int_0^{2\pi} e^{-i(j_r-k_r)\theta_r} d\theta_r.$$

Thus, $\tilde{\sigma}_0(\psi_{j,k}) = 0$ when $j \neq k$. Moreover, denoting $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^d$,

$$\begin{aligned}
\tilde{\sigma}_0(\psi_{k,k}) &= (2\pi)^{\frac{d}{2}}(-1)^k 2^d \int_{\mathbb{R}^d_+} \sigma(2\rho_1^2, \ldots, 2\rho_d^2) L_k(2\rho_1^2, \ldots, 2\rho_d^2) e^{-|\rho|^2} \rho^{\mathbf{1}} d\rho \\
&= (2\pi)^{\frac{d}{2}}(-1)^k 2^{-d} \int_{\mathbb{R}^d_+} \sigma(y) \mathcal{L}_k(y) dy = \sigma_k
\end{aligned}$$

By (4.11), with $g_k = \langle g, h_k \rangle$ and $f_k = \langle f, h_k \rangle$, we obtain

$$(W_{\tilde{\sigma}_0} f)(g) = (2\pi)^{-\frac{d}{2}} \sum_{k \in \mathbb{N}_0^d} g_k f_k \sigma_k. \tag{4.12}$$

Let now $f, g \in \mathcal{S}'(\mathbb{R}^d)$. Define

$$(W_{\tilde{\sigma}_0} f)(g) = \sum_{n \in \mathbb{N}_0^d} f_n g_n \sigma_n.$$

Observe that the series is absolutely convergent since $\{f_n\}_{n \in \mathbb{N}_0^d}, \{g_n\}_{n \in \mathbb{N}_0^d} \in s'$ and $\{\sigma_n\}_{n \in \mathbb{N}_0^d} \in s$ ($\sigma \in \mathcal{S}(\mathbb{R}^d_+)$; cf. Theorem 2.2.1). Thus, if we fix $f \in \mathcal{S}'(\mathbb{R}^d)$, the mapping

$$g \mapsto (W_{\tilde{\sigma}_0} f)(g),\ \mathcal{S}'(\mathbb{R}^d) \to \mathbb{C}$$

is well defined linear mapping.

To prove that it is continuous let B be a bounded subset of $\mathcal{S}'(\mathbb{R}^d)$. As $\mathcal{S}(\mathbb{R}^d)$ is barreled B is equicontinuous. Thus, the set $\{\{g_k\}_{k \in \mathbb{N}_0^d} | \, g \in B\}$ is equicontinuous subset of s'. We conclude that there exist $r \in \mathbb{N}$ and $C > 0$ such that

$$|g_k| \leq C(|k|+1)^r,\ \forall k \in \mathbb{N}_0^d,\ \forall g \in B.$$

Hence,

$$\sup_{g \in B} |(W_{\tilde{\sigma}_0} f)(g)| \leq C \sum_{n \in \mathbb{N}_0^d} |f_n|(|n|+1)^r |\sigma_n| < \infty,$$

i.e. $W_{\tilde{\sigma}_0} f$ maps bounded subsets in $\mathcal{S}'(\mathbb{R}^d)$ into bounded subsets of $\mathbb{C}$. Since $\mathcal{S}'(\mathbb{R}^d)$ is bornological,

$$g \mapsto (W_{\tilde{\sigma}_0} f)(g)$$

is continuous. Hence $W_{\tilde{\sigma}_0} f \in \mathcal{S}(\mathbb{R}^d)$ ($\mathcal{S}(\mathbb{R}^d)$ is reflexive). Now we can easily conclude that

$$W_{\tilde{\sigma}_0} f = \sum_{n \in \mathbb{N}_0^d} f_n \sigma_n h_n.$$

Thus, the mapping

$$f \mapsto W_{\tilde{\sigma}_0} f,\ \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),$$

is well defined and linear. Arguing similarly as before, one can prove that when f varies in a bounded subset B of $\mathcal{S}'(\mathbb{R}^d)$ the set

$$\{\{f_k\sigma_k\}_{k\in\mathbb{N}_0^d} | \, f \in B\} \text{ is a bounded subset of } s.$$

Thus,

$$\{W_{\tilde{\sigma}_0} f | \, f \in B\} \text{ is bounded subset of } \mathcal{S}(\mathbb{R}^d).$$

As $\mathcal{S}'(\mathbb{R}^d)$ is bornological, the mapping

$$f \mapsto W_{\tilde{\sigma}_0} f, \ \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),$$

is continuous. Observe that $W_{\tilde{\sigma}_0} f$ coincides with the Weyl transform of f when $f \in \mathcal{S}(\mathbb{R}^d)$ (cf. (4.12)).

If $\sigma_j \to \sigma$ as $j \to \infty$, in $\mathcal{S}(\mathbb{R}_+^d)$, Theorem 2.2.1 implies that

$$\{\sigma_{n,j}\}_{n\in\mathbb{N}_0^d} \xrightarrow{s} \{\sigma_n\}_{n\in\mathbb{N}^d} \text{ as } j \to \infty.$$

Thus for each fixed $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$\sum_{n\in\mathbb{N}_0^d} f_n \sigma_{n,j} h_n \xrightarrow{\mathcal{S}(\mathbb{R}^d)} \sum_{n\in\mathbb{N}_0^d} f_n \sigma_n h_n,$$

i.e. $W_{\tilde{\sigma}_{0,j}} \to W_{\tilde{\sigma}_0}$ in the topology of simple convergence in $\mathcal{L}(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. Now, the Banach-Steinhaus theorem implies that the convergence holds in the topology of precompact convergence. Since $\mathcal{S}'(\mathbb{R}^d)$ is Montel, the convergence also holds in the strong topology of $\mathcal{L}(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$. □

Theorem 4.5.2. *Let σ be a measurable function on $\mathbb{R}_+^d$ such that there exists $n \in \mathbb{N}_0^d$ for which*

$$\frac{\sigma(\rho)}{(\mathbf{1}+\rho)^n} \in L^2(\mathbb{R}_+^d).$$

We have

$$\tilde{\sigma}(x,\xi) = \sigma(2v(x,\xi)) \in (\mathcal{S}(\mathbb{R}^{2d}))'.$$

Then

$$W_{\tilde{\sigma}} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$$

is continuous mapping and it extends to a continuous mapping

$$(\mathcal{S}(\mathbb{R}^d))' \to (\mathcal{S}(\mathbb{R}^d))'.$$

Let $\sigma^{(j)}$, $j \in \mathbb{N}$, be measurable functions on $\mathbb{R}_+^d$ such that for each $j \in \mathbb{N}$ there exists $n^{(j)} \in \mathbb{N}_0^d$ for which

$$\sigma_j(\rho)/(\mathbf{1}+\rho)^{n^{(j)}} \in L^2(\mathbb{R}_+^d).$$

If σ is a measurable function on $\mathbb{R}_+^d$ with the properties stated above and if

$$\sigma^{(j)}(\eta) \xrightarrow{(\mathcal{S}(\mathbb{R}_+^d))'} \sigma(\eta) \qquad \text{as } j \to \infty,$$

then $W_{\tilde{\sigma}^{(j)}} \to W_{\tilde{\sigma}}$ in the strong topology of not only $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$ but also $\mathcal{L}((\mathcal{S}(\mathbb{R}^d))', (\mathcal{S}(\mathbb{R}^d))')$.

Proof. Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Denote

$$f_k = \langle f, h_k\rangle,\ g_k = \langle g, h_k\rangle \text{ and } \sigma_k = (2\pi)^{d/2}(-1)^k 2^{-d}\langle \sigma, \mathcal{L}_k\rangle.$$

Following (4.4), we obtain

$$\begin{aligned}(W_{\tilde\sigma(x,\xi)}f)(g) &= (2\pi)^{-d/2}\tilde\sigma(W(f,\overline{g}))\\ &= (2\pi)^{-d/2}\sum_{k\in\mathbb{N}_0^d}\sum_{j\in\mathbb{N}_0^d}\langle g, h_k\rangle\langle f, h_j\rangle\tilde\sigma(\psi_{j,k}) \end{aligned}\tag{4.13}$$

where $\psi_{j,k} = W(h_j, h_k)$. Next,

$$\tilde\sigma(\psi_{j,k}) = \langle\tilde\sigma(x,\xi), \psi_{j,k}(x,\xi)\rangle = \sum_{|s|\le N} 2^{|s|}\int_{\mathbb{R}^{2d}} v(x,\xi)^s\sigma(2v(x,\xi))\psi_{j,k}(x,\xi)dxd\xi$$

By the same technique as in the proof of Theorem 4.5.1 we have

$$\tilde\sigma(\psi_{j,k}) = \sum_{|s|\le N} C_{j,k,s}\prod_{r=1}^{d}\int_0^{2\pi} e^{-i(j_r-k_r)\theta_r}d\theta_r.$$

Thus $\tilde\sigma(\psi_{j,k}) = 0$ for $j \neq k$. Moreover, denoting $\mathbf{1} = (1,\dots,1) \in \mathbb{N}^d$,

$$\begin{aligned}\tilde\sigma(\psi_{k,k}) &= (2\pi)^{d/2}(-1)^k 2^d\int_{\mathbb{R}_+^d}\sum_{|s|\le N}2^{|s|}(\rho^s)^2\sigma_s(2\rho_1^2\dots,2\rho_d^2)\\ &\qquad\times L_k(2\rho_1^2,\dots,2\rho_d^2)e^{-|\rho|^2}\rho^{\mathbf{1}}d\rho\\ &= (2\pi)^{d/2}(-1)^k 2^{-d}\langle\sigma, \mathcal{L}_k\rangle = \sigma_k.\end{aligned}$$

By (4.13), with $g_k = \langle g, h_k\rangle$, $f_k = \langle f, h_k\rangle$ we obtain

$$(W_{\tilde\sigma}f)(g) = (2\pi)^{-d/2}\sum_{k\in\mathbb{N}_0^d} g_k f_k\sigma_k.$$

Since $f, g \in \mathcal{S}(\mathbb{R}^d)$ it follows that $\{f_k\}_{k\in\mathbb{N}_0^d} \in s$ and $\{g_k\}_{k\in\mathbb{N}_0^d} \in s$. As $\sigma \in \mathcal{S}'(\mathbb{R}_+^d)$, applying Theorem 2.3.1 we obtain $\{\sigma_k\}_{k\in\mathbb{N}_0^d} \in s'$. This implies that the series is absolutely convergent. If B is a bounded subset of $\mathcal{S}(\mathbb{R}^d)$ than

$$\{\{g_k\}_{k\in\mathbb{N}_0^d}|\, g \in B\} \text{ is bounded subset of } s.$$

Thus,

$$\{(W_{\tilde\sigma}f)(g)|\, g \in B\} \text{ is bounded subset of } \mathbb{C}.$$

Hence, the mapping

$$g \mapsto (W_{\tilde\sigma}f)(g),\ \mathcal{S}(\mathbb{R}^d) \to \mathbb{C},$$

is continuous since $\mathcal{S}(\mathbb{R}^d)$ is bornological. Thus,

$$f \mapsto W_{\tilde\sigma}f,\ \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d),$$

is a well defined linear mapping. Observe that for each $n \in \mathbb{N}_0^d$, $(W_{\tilde{\sigma}}f)(h_n) = \sigma_n f_n$. Since $\{\sigma_n\}_{n\in\mathbb{N}_0^d} \in s'$ and $\{f_n\}_{n\in\mathbb{N}_0^d} \in s$, we have $\{\sigma_n f_n\}_{n\in\mathbb{N}_0^d} \in s$, i.e. $W_{\tilde{\sigma}}f \in \mathcal{S}(\mathbb{R}^d)$. We conclude that

$$f \mapsto W_{\tilde{\sigma}}f,\ \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),$$

is well defined linear map. Moreover,

$$W_{\tilde{\sigma}}f = \sum_{n\in\mathbb{N}_0^d} \sigma_n f_n h_n.$$

To prove that it is continuous let B be a bounded subset of $\mathcal{S}(\mathbb{R}^d)$. As $\{\sigma_k\}_{k\in\mathbb{N}^d} \in s'$, the set

$$\{\{\sigma_n f_n\}_{n\in\mathbb{N}_0^d} | \, f \in B\} \text{ is bounded in } s.$$

Thus,

$$\{W_{\tilde{\sigma}}f | \, f \in B\} \text{ is bounded in } \mathcal{S}(\mathbb{R}^d).$$

As $\mathcal{S}(\mathbb{R}^d)$ is bornological,

$$f \mapsto W_{\tilde{\sigma}}f,\ \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d),$$

is continuous. By similar technique, one proves that

$$W_{\tilde{\sigma}}f = \sum_{n\in\mathbb{N}_0^d} f_n s_n h_n \in \mathcal{S}'(\mathbb{R}^d), \text{ for each } f \in \mathcal{S}'(\mathbb{R}^d)$$

and the mapping,

$$f \mapsto W_{\tilde{\sigma}}f,\ \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d),$$

is continuous.

Let $\sigma, \sigma^{(j)} \in \mathcal{S}'(\mathbb{R}_+^d)$, $j \in \mathbb{N}$, be as in the assumption in the Theorem, with $\sigma^{(j)} \to \sigma$ in $\mathcal{S}'(\mathbb{R}_+^d)$. In order to prove $W_{\tilde{\sigma}^{(j)}} \to W_{\tilde{\sigma}}$ in the strong topology of $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d))$ (resp. $\mathcal{L}(\mathcal{S}'(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$) it is enough to prove that for each $f \in \mathcal{S}(\mathbb{R}^d)$ (resp. $f \in \mathcal{S}'(\mathbb{R}^d)$), $W_{\tilde{\sigma}^{(j)}}f \to W_{\tilde{\sigma}}f$ in $\mathcal{S}(\mathbb{R}^d)$ (resp. $W_{\tilde{\sigma}}f \in \mathcal{S}'(\mathbb{R}^d)$) since in this case the Banach-Steinhaus theorem implies convergence in the topology of precompact convergence. As $\mathcal{S}(\mathbb{R}^d)$ (resp. $\mathcal{S}'(\mathbb{R}^d)$) is Montel the convergence also holds in the strong topology. Thus fix $f \in \mathcal{S}(\mathbb{R}^d)$ (resp. $f \in \mathcal{S}'(\mathbb{R}^d)$). Theorem 2.3.1 implies that

$$\{\sigma_k^{(j)}\}_{k\in\mathbb{N}_0^d} \xrightarrow{s'} \{\sigma_k\}_{k\in\mathbb{N}_0^d}.$$

But then

$$\{\sigma_k^{(j)} f_k\}_{k\in\mathbb{N}_0^d} \xrightarrow{s} \{\sigma_k f_k\}_{k\in\mathbb{N}_0^d}, \text{ (resp. in } s')$$

i.e.

$$W_{\tilde{\sigma}^{(j)}}f \xrightarrow{\mathcal{S}(\mathbb{R}^d)} W_{\tilde{\sigma}}f \text{ (resp. in } \mathcal{S}(\mathbb{R}^d)).$$

□

Appendix A

Topics from Functional Analysis

A.1 Nuclear Mappings

Note that in each linear space E there is a one to one relation between seminorms and central subsubsets (absorbing absolutely convex subset A of a linear space E is central if $x \in A$ whenever $\alpha x \in A$ for all $\alpha \in \mathbb{C}$ with $|\alpha| < 1$). For each such set A, the equation

$$p_A(x) = \inf\{\varrho > 0 : x \in \varrho A\} \qquad \text{for } x \in E$$

determines a semi-norm p_A for which

$$A = \{x \in E : p_A(x) \leq 1\}.$$

Conversely, every semi-norm p can be obtained in this way from the central subset

$$A = \{x \in E : p(x) \leq 1\}.$$

We shall consider nuclear mappings from a normed space E into a normed space F.

Definition A.1.1. ([30, Definition 3.1.1, p.49]) Let E and F be two arbitrary normed spaces with closed unit balls U and V. A continuous linear mapping $T : E \to F$ is called nuclear if there are continuous linear forms $a_n \in E'$ and elements $y_n \in F$ with

$$\sum_n p_{U^\circ}(a_n) p_V(y_n) < \infty$$

such that T has the form

$$Tx = \sum_n \langle x, a_n \rangle y_n \qquad \text{for } x \in E.$$

For each nuclear mapping T we set

$$\nu(T) = \inf\{\sum_n p_{U^\circ}(a_n) p_V(y_n)\},$$

where the infimum is taken over all possible representations of T. If F is continuously injected into a larger normed space G it is possible that the continuous linear mapping $T : E \to F$ is nuclear as a mapping from E into G but not as a mapping from E into F. The detailed investigation of these matters led to the concept of a quasinuclear mapping.

Definition A.1.2. ([30, Definition 3.2.3, p.56]) If E and F are two arbitary normed spaces with closed unit balls U and V, then we designate a continuous linear mapping T from E into F as quasinuclear if there is a sequence of linear forms $a_n \in E'$ with

$$\sum_n p_{U^\circ}(a_n) < \infty$$

such that

$$p_V(Tx) \leq \sum_n |\langle x, a_n \rangle| \qquad \text{for } x \in E.$$

For each quasinuclear mapping T we set

$$\pi_0(T) = \inf\{\sum_n p_{U^\circ}(a_n)\},$$

where the infimum is taken over all sequences of linear forms a_n which have the stated property.

Proposition A.1.1. *([30, Proposition 3.2.7, p.59]) A continuous linear mapping T from a normed space E into a normed space F is quasinuclear if and only if there is a normed space G containing F such that T is nuclear as a mapping from E into G.*

For E, F and G three normed spaces we have the following theorem:

Theorem A.1.1. *[30, Proposition 3.3.2, p.62]) Let $T : E \to F$ and $S : F \to G$ be two quasinuclear mappings. Then the product ST is nuclear and*

$$\nu(ST) \leq \pi_0(S)\pi_0(T).$$

We proceed to define the nuclear space.

Let $\hat{X}_p$ be the completion of the normed space $E/\text{Ker }p$ (the latter is a normed space of we put on it the quotient mod Ker p of the seminorm p).

Definition A.1.3. (Nuclear space) A locally convex Hausdorff topological vector space X is called nuclear if to every continuous seminorm p on X there is another continuous seminorm on X, $q \geq p$, such that the canonical mapping $\hat{X}_q \to \hat{X}_p$ is nuclear.

A.2 The Open Mapping Theorem

Suppose f maps S into T where S and T are topological spaces. We say that f is open at a point $p \in S$ if $f(V)$ contains a neighborhood of $f(p)$ whenever V is a neighborhood of p. We say that f is open if $f(U)$ is open in T whenever U is open in S.

It is clear that f is open if and only if f is open at every point in S. Because of the invariance of vector topologies, it follows that a linear mapping of one topological vector space into another is open if and only if it is open at the origin.

A continuous linear mapping f of S onto T is called a topological homomorphism if it is open. Note that if f is also one-to-one f is called a topological monomorphism. f is then a homeomorphism of S and $f(S)$. If $f(S) = T$ as well, f is called a topological isomorphism of S and T (see [22, p.91]).

Now we state the open mapping theorem:

Theorem A.2.1. *([35, Theorem 2.11, p.47]) Suppose*

(a) X *is an* (F)*-space,*

(b) Y *is a topological vector space,*

(c) $\Lambda : X \to Y$ *is continuous and linear and*

(d) $\Lambda(X)$ *is of the second category in* Y.

Then

(i) $\Lambda(X) = Y$,

(ii) Λ *is an open mapping and*

(iii) Y *is an* (F)*-space.*

Corollary A.2.1. *([35, Corollary 2.12, p.48])*

(i) If Λ *is a continuous linear mapping of an* (F)*-space* X *onto an* (F)*-space* Y*, then* Λ *is open.*

(ii) If Λ *satisfies* (i) *and is one-to-one, then* $\Lambda^{-1} : Y \to X$ *is continuous.*

A.3 The closed-graph theorem

If X and Y are sets and f maps X into Y, the graph of f is the set of all points $(x, f(x))$ in the cartesian product $X \times Y$.

Proposition A.3.1. *([35, Proposition 2.14, p.49]) If* X *is a topological space and if* Y *is a Hausdorff space, and* $f : X \to Y$ *is continuous, then the graph* G *of* f *is closed.*

Now we state the closed graph theorem:

Theorem A.3.1. *([35, Theorem 2.15, p.50]) Suppose*

(a) X *and* Y *are* (F)*-spaces,*

(b) $\Lambda : X \to Y$ *is linear,*

(c) $G = \{(x, \Lambda x) : x \in X\}$ *is closed in* $X \times Y$.

Then Λ *is continuous.*

A.4 Topology of bounded convergence

Let E and F be two locally convex topological vector spaces. By $\mathcal{L}(E,F)$ we denote the collection of all continuous linear mappings from E into F.

We are interested in the family of all bounded sets of E, which leads to the topology of bounded convergence; equipped with it, $\mathcal{L}(E,F)$ will be denoted by $\mathcal{L}_b(E,F)$. Also, $\mathcal{L}_b(E,\mathbb{C}) = E'_b$, strong dual of E.

Corollary A.4.1. *([39, Corollary 1, p. 344]) Let E be a locally convex Hausdorff space such that a linear mapping of E into a locally convex space which is bounded on every bounded set is continuous. Then for all complete locally convex Hausdorff spaces F, $\mathcal{L}_b(E,F)$ is complete. In particular, E'_b is complete.*

A.5 Tensor product

In this chapter we review the topological tensor products. For more details on this subject we refer to [39].

We begin with the definition of the tensor product of two vector spaces.

Definition A.5.1. (Algebraic tensor product) Let E and F be two vector spaces over $\mathbb{K} = \{\mathbb{R}, \mathbb{C}\}$. We form the set $\Lambda(E\times F)$ of all formal finite linear combinations

$$\sum_{(x,y)\in E\times F} (x,y)\alpha_{x,y}$$

of elements of $E \times F$, with coefficients in $\mathbb{K}$. $\Lambda(E\times F)$ becomes a vector space over $\mathbb{K}$ when we put

$$\Big(\sum_{(x,y)\in E\times F} (x,y)\alpha_{x,y}\Big)\beta = \sum_{(x,y)\in E\times F} (x,y)\alpha_{x,y}\beta$$

and

$$\sum_{(x,y)\in E\times F} (x,y)\alpha_{x,y} + \sum_{(x,y)\in E\times F} (x,y)\beta_{x,y} = \sum_{(x,y)\in E\times F} (x,y)(\alpha_{x,y} + \beta_{x,y}).$$

The zero element is obtained when all the coefficients $\alpha_{x,y}$ are put to be equal to 0.

We now form the linear span Λ_0 in $\Lambda(E\times F)$ of all elements of the form

$$\Big(\sum_{i=1}^{n} x_i\alpha_i, \sum_{k=1}^{m} y_k\beta_k\Big) - \sum_{i=1}^{n}\sum_{k=1}^{m}(x_i, y_k)\alpha_i\beta_k.$$

The quotient space Λ/Λ_0 is called the tensor product, $E\otimes F$ of E and F.

A mapping $B(x,y)$ from $E\times F$ into a vector space H which is linear in both variables i.e. is called a bilinear mapping from $E\times F$ into H. Thus, for all $x_i \in E$ and $y_k \in F$,

$$B(\sum_i x_i\alpha_i, \sum_k y_k\beta_k) = \sum_i\sum_k B(x_i, y_k)\alpha_i\beta_k.$$

If H is the field of coefficients, we speak of bilinear forms or bilinear functionals. The set of all bilinear mappings from $E \times F$ into H form a vector space $\mathcal{B}(E \times F, H)$. We denote the space of all bilinear forms on $E \times F$ by $\mathcal{B}(E \times F)$.

We can topologize and form the completion of a tensor product $E \otimes F$ either by relying directly on the seminorms on E and F or by embedding $E \otimes F$ in some space related to E and F in which a "natural" topology already exists. The first approach leads to the so-called projective or π-topology. The second approach leads to a variety of topologies, the most important is the ε-topology. We proceed to give the definition of the firs main topology on tensor products.

Definition A.5.2. (Projective tensor product) Let $X \otimes Y$ be the algebraic tensor product of locally convex spaces X and Y. The projective tensor topology or the π-topology of $X \otimes Y$ is the strongest topology for which the bilinear mapping

$$((x, y) \mapsto x \otimes y) : X \times Y \to X \otimes Y$$

is continuous. This topological space is denoted by $X \otimes_\pi Y$ and its completion by $X \hat{\otimes}_\pi Y$.

The definition on the ε-topology is based on the relationship between tensor products and bilinear functionals.

Proposition A.5.1. *([39, Proposition 42.4., p.432]) Let X, Y be locally convex spaces over $\mathbb{C}$. The algebraic tensor product $X \otimes Y$ is isomorphic to the space $B(X', Y')$ of continuous bilinear functionals $X' \times Y' \to \mathbb{C}$, where X' and Y' are the dual spaces with weak topologies.*

We introduce the notion of equicontinuous sets of functions:

Definition A.5.3. (Equicontinuity in vector space) Let X be a topological space and V a topological vector space. A familly $\mathcal{F}$ of mappings $f : X \to V$ is called equicontinuous at $p \in X$ if for every neighborhood $W \subset V$ of $f(p)$ there exists a neighborhood $U \subset X$ of p such that $f(x) \in W$ whenever $f \in \mathcal{F}$ and $x \in U$.

Now we state the definition of the second main topology on tensor products.

Definition A.5.4. (Injective tensor product) Let X, Y be locally convex spaces over $\mathbb{C}$. Let $\tilde{B}(X', Y')$ be the space of those bilinear functionals $X' \times Y' \to \mathbb{C}$ that are continuous separately in each variable. Endow $\tilde{B}(X', Y')$ with the topology τ of uniform convergence on the products of an equicontinuous subset of X' and an equicontinuous subset of Y'. Interpreting $X \otimes Y \subset \tilde{B}(X', Y')$ as in Proposition A.5.1, let the injective tensor topology or ε-topology be the restriction of τ to $X \otimes Y$. This topological space is denoted by $X \otimes_\varepsilon Y$ and its completion by $X \hat{\otimes}_\varepsilon Y$.

The introduction of nuclear spaces is justified by the following theorem (for the definition of the nuclear spaces see Appendix A.1):

Theorem A.5.1. *([39, Theorem 50.1. (f), p.511]) X is nuclear if and only if for every locally convex Hausdorff topological vector space Y, the canonical mapping of $X \hat{\otimes}_\pi Y$ into $X \hat{\otimes}_\varepsilon Y$ is an isomorphism onto.*

Note that the previous theorem means that

$$X \otimes_\pi Y = X \otimes_\varepsilon Y,$$

where the equality extends to the topologies.

Proposition A.5.2. *([39, Proposition 50.1. (50.9), p.514]) If X and Y are two nuclear spaces, $X\hat{\otimes}Y$ is nuclear.*

We close this section with the results about Fréchet spaces.

Proposition A.5.3. *([39, Proposition 50.6., p.523]) A Fréchet space X is nuclear if and only if its strong dual is nuclear.*

Proposition A.5.4. *([39, Proposition 50.7., p.524]) Let E and F be two Fréchet spaces. If E is nuclear we have the canonical isomorphism*

$$E'\hat{\otimes}F' \cong B(E,F) \cong (E\hat{\otimes}F)'.$$

A.6 Barreled and Montel spaces

Among the locally convex topologies on E_1 which can be defined in terms of the dual pair $\langle E_2, E_1\rangle$ there is a finest one, namely the topology of uniform convergence on all the weakly bounded subsets of E_2. This is called the strong topology $\mathcal{I}_b(E_2)$ on E_1. We now give the characterization of the strong topology.

A subset T of a locally convex space $T[\mathcal{I}]$ is called a barrel if T has the following properties:

(i) T is absorbent (A subset M of E is said to be absorbent if a suitable multiple ρx, $\rho > 0$, of each element x of E lies in M);

(ii) T is closed;

(iii) T is absolutely convex.

A locally convex space is said to be barreled if the barrels form a base of $\mathcal{I}$-neighborhoods of $\circ$ (see [22, p. 257]). This is equivalent to: the barreled spaces $E[\mathcal{I}]$ are those locally convex spaces whose topology $\mathcal{I}$ coincides with the strong topology $\mathcal{I}_b(E')$ (see [22, §21, 2.(2), p.257]).

Corollary A.6.1. *([22, §21, 5.(3), p.263]) All (F)-spaces are barreled.*

The importance of barreled spaces stems mainly from the following result:

Theorem A.6.1. *([39, Theorem 33.1.,p.347]) Let E be barreled and F a locally convex space. The following properties of a subset H of the space $\mathcal{L}(E,F)$ of continuous linear mappings of E into F are equivalent:*

(i) H is bounded for the topology of pointwise convergence;

(ii) H is bounded for the topology of bounded convergence;

(iii) H is equicontinuous.

Now, we define a filter $\mathcal{F}$ is a family of subsets in E, submitted to three conditions:

(F_1) The empty set should not belong to the family $\mathcal{F}$.

(F_2) The intersection of any two sets, belonging to the family, also belongs to the family $\mathcal{F}$.

(F_3) Any set, which contains the set belonging to $\mathcal{F}$, should also belong to $\mathcal{F}$.

The theorem which follows is often referred to as the Banach-Steinhaus theorem.

Theorem A.6.2. *([39, p. 348]) Let E be a barreled space, F a locally convex Hausdorff space and $\mathcal{F}$ a filter on $\mathcal{L}(E, F)$ which converges pointwise in E to a linear map u_0 of E into F. Suppose that $\mathcal{F}$ has either one of the following two properties:*

(i) There is a set H, belonging to $\mathcal{F}$,which is bounded for the topology of pointwise convergence.

(ii) $\mathcal{F}$ has a countable basis.

The u_0 is a continuous linear mapping of E into F and $\mathcal{F}$ converges to u_0 uniformly on every compact subset of E.

There is a class of barreled spaces which is of particular interest. A barreled space $E[\mathcal{I}]$ is called a Montel space or (M)-space if every bounded subset of E is relatively compact (see [22, p. 369]). It follows from the definition that:

Proposition A.6.1. *([22, §27, 2.(1), p.369]) Every M-space is reflexive.*

Moreover, the strong and weak topologies on the dual of an (M)-space coincide.

Proposition A.6.2. *([22, §27, 2.(2), p.369]) The strong dual of an M-space is again an M-space.*

Thus, the weak and the strong topologies coincide on the bounded subsets of an M-space. In particular, we have

Proposition A.6.3. *Every weakly convergent sequence in an M-space is also strongly convergent, to the same limit.*

We give the continuity Theorem for bilinear mappings on barreled (DF)-spaces.

Theorem A.6.3. *([23, §40, 2.(11)]) Let E, F be barreled (DF)-spaces, G locally convex. Then every $B \in \mathcal{B}(E \times F, G)$ is continuous.*

A.7 Bornological and ultrabornological spaces

A linear functional on a normed space is continuous if it is bounded on the unite ball. This can also be expressed by saying that every linear functional on a normed space which is bounden on the bounded sets is continuous.

Expressed in this form,this property need no longer hold for arbitrary locally convex spaces. If we say that a linear functional $u \in E^*$ is locally bounden when its values remain bounded on any bounden subset of E, then the problem is to characterize those locally convex spaces for which every locally bounded linear functional is continuous. For this we can always suppose that the topology $\mathcal{I}$ is the Mackey topology (i.e. the toplogy $E[\mathcal{I}_k(E')]$. That is the topology of uniform convergence on all absolutely convex weakly compact subsets of E' is a locally convex topology on E which is finer then the original topology (see [22, p. 260])).

In order to give the answer to this question we need the following definition: A locally convex space $E[\mathcal{I}]$ is said to be bornological if every absolute convex set M which absorbs all the bounded sets of $E[\mathcal{I}]$ is a $\mathcal{I}$-neighborhood of $\circ$ (see [22, p. 379]).

Now, the problem stated at the beginning of this section is answered by the following proposition.

Proposition A.7.1. *([22, §28, 1.(3), p.379]) A locally convex space $E[\mathcal{I}]$ has the property that every locally bounded linear functional on E is continuous if and only if $E[\mathcal{I}_k(E')]$ is bornological.*

The structure of bornological spaces is given by:

Proposition A.7.2. *([22, §28, 2.(2), p.381]) Every bornological space is the locally convex hull $E[\mathcal{I}] = \sum_B E_B$ of normed spaces E_B. If, further, $E[\mathcal{I}]$ is sequentially complete, $E[\mathcal{I}]$ is the locally convex hull of B-spaces.*

Next, we state the results which generalizes previous proposition.

Theorem A.7.1. *([22, §28, 2.(3), p.381]) A locally convex space $E[\mathcal{I}]$ is bornological if and only if every locally bounded map from $E[\mathcal{I}]$ into any locally convex space $F[\mathcal{I}']$ is continuous.*

Criterion for the continuity of linear mappings from bornological spaces can be expressed in another form which is particular convinient for applications.

Theorem A.7.2. *([22, §28, 3.(4), p.383]) A linear mapping A from a bornological space into a locally convex space is continuous if and only if it is sequentially continuous and if and only if A is locally bounded.*

The class of bornological space is stable under various operations.

Proposition A.7.3. *([22, §28, 4.(1), p.383]) Every locally convex hull of bornological spaces is bornological.*

Proposition A.7.4. *([22, §28, 4.(4), p.384]) The topological product of at most countably many bornological space is again bornological.*

Since every metrizable locally convex space is bornological, we otain a very extensive class of bornological spaces by repeating application of Proposition A.7.3 and Proposition A.7.4. For example, all (LF)-spaces i.e. a locally convex spaces that can be represented as the topological inductive limit of a properly increasing sequences $E_1[\mathcal{I}_1] \subset E_2[\mathcal{I}_2] \subset \cdots$ of (F)-spaces, belong to this class.

If a vector space E is the linear span of certain linear subspaces E_α, we write $E = \sum_\alpha E_\alpha$. Of particular interest for us is the case where each E_α is given as a linear image $A_\alpha(F_\alpha)$ of a vector space F_α. We then write $E = \sum_\alpha A_\alpha(F_\alpha)$.

A special case of such a linear span is the direct sum $E = \oplus_\alpha E_\alpha$.

If the F_α are locally convex topological vector space $F_\alpha[\mathcal{I}_\alpha]$, we can try to introduce a locally convex topology on the linear span $E = \sum_\alpha A_\alpha(F_\alpha)$. By analogy with the spacial case of the locally convex direct sum, the finest locally convex topology $\mathcal{I}$ for which all the A_α are continuous mappings from F_α into E suggests itself. This topology $\mathcal{I}$ need not however be Hausdorff. But if this is the case $E = \sum_\alpha A_\alpha(F_\alpha[\mathcal{I}_\alpha])$ is called the locally convex hull of the $A_\alpha(F_\alpha[\mathcal{I}_\alpha])$ and $\mathcal{I}$ is called the hull topology on E (see [22, p. 215]).

A locally convex topological vector space E is called ultrabornological if it can be represented as the locally convex hull $E = \sum_\alpha A_\alpha(E_\alpha)$ of (B)-spaces E_α. E always has a representation of the simpler form $E = \sum_\alpha F_\alpha$ where the F_α are again (B)-spaces. By Proposition A.7.3 every ultrabornological space is bornological. Conversely, every sequentially complete bornological space is ultrabornological by Proposition A.7.2 (see [23, p. 43]).

We have the following theorem

Theorem A.7.3. *([23, §34, 8.(6), p.44]) Every closed linear mapping of an ultrabornological space into an LF-space is continuous.*

Every continuous linear mapping of an LF-space onto an ultrabornological space is a homomorphism.

A.8 De Wilde's theory

Theorem A.7.3 should be true for a much larger class of spaces than the class of (LF)-spaces. Let E be the class of ultrabornological spaces, a subclass of the class of barreled spaces, and we are looking for spaces F such that the closed-graph theorem for mapping from any E into F is true. We give here an exposition of De Wilde's approach (see [23, p. 53]).

We start with the fundamental notion of a web in a locally convex space E. Let $\mathcal{W} = \{C_{n_1,\ldots,n_k}\}$ be a class of subsets $C_{n_1,\ldots,n_k}$ of E, where k and $n_1, \ldots, n_k$ run through all the natural numbers. $\mathcal{W}$ is called the web if it satisfies the relationships

$$E = \bigcup_{n_1=1}^{\infty} C_{n_1} \quad \text{and} \quad C_{n_1,\ldots,n_{k-1}} = \bigcup_{n_1=1}^{\infty} C_{n_1,\ldots,n_k}$$

for $k > 1$ and all $n_1, \ldots, n_{k-1}$. If all the sets of a web are closed or absolutely convex, we say that the web is closed resp. absolutely convex.

A $\mathcal{W}$ is a $\mathcal{C}$-web if the following condition is satisfied: for every fixed sequence

n_k, $k = 1, 2, ...$, there exists a sequence of positive numbers ρ_k such that for all λ_k, $0 \leq \lambda_k \leq \rho_k$ and all $x_k \in C_{n_1,\ldots,n_k}$the series $\sum_{k=1}^{\infty} \lambda_k x_k$ converges in E.

A locally convex topological vector space $E[\mathcal{I}]$ in which there exists a $\mathcal{C}$-web will be said to be a webbed space.

Now, let E and F be locally convex; a linear mapping A of E into F is called sequentially closed if its graph $G(A)$ is sequentially closed in $E \times F$ (A set $E \subseteq X$ is said to be sequentially closed if for every sequence $\{x_i\}_{i=1}^{\infty}$ of elements of E and every $x \in X$ such that $\{x_i\}_{i=1}^{\infty}$ converges to x in X we have that $x \in E$). In view of applications it is certainly important to have the closed-graph theorem in the stronger form that A is continuous if it is only sequentially closed. We obtain De Wilde's closed-graph theorem for ultrabornological spaces:

Theorem A.8.1. *([23, §35, 2.(2), p. 57]) A sequentally closed linear mapping of an ultrabornological space E into a webbed space F is continuous.*

The classes of webbed spaces are stable under:

Proposition A.8.1. *([23, §35, 4.(8), p. 63]) The topological inductive limit $E[\mathcal{I}] = \varinjlim_{n\to\infty} E_n[\mathcal{I}_n]$ of a sequence webbed spaces $E_n[\mathcal{I}_n]$ is of the same type.*

The question whether the strong dual of a webbed space is again webbed seems to be open. But there are some results in this direction.

Proposition A.8.2. *([23, §35, 4.(11), p. 64]) The strong dual of a metrizable space E is strictly webbed.*

We conclude this section with some remark on the hereditary property of ultrabornological spaces.

Proposition A.8.3. *([23, §35, 7.(7), p. 72]) The locally convex hull $E[\mathcal{I}] = \sum_\alpha A_\alpha(E_\alpha[\mathcal{I}_\alpha])$ of ultrabornological spaces $E_\alpha[\mathcal{I}_\alpha]$ ia an ultrabornological spaces.*

Appendix B

Komatsu's approach to ultradistributions

We follow H. Komatsu (see [21]).

Let $\{M_p\}_{p\in\mathbb{N}_0}$ be a sequence of positive numbers. An infinitely diferentiable function f on an open set Ω in $\mathbb{R}^d$ is called an ultradifferentiable function of class M_p if on each compact set K in Ω its derivatives are estimated in the form

$$\|D^\alpha f\|_{C(K)} \le Ch^{|\alpha|}M_{|\alpha|},\ |\alpha| = 0, 1, \dots\ .$$

We call f an ultradifferentiable function of class $\{M_p\}$ if the above inequality holds for some $h > 0$.

We impose the following conditions on M_p

$(M.1)$ (logarithmic convexity)

$$M_p^2 \le M_{p-1}M_{p+1},\ p \in \mathbb{N}.$$

$(M.2)$ (stability under ultradifferential operators) There are constants A and H such that

$$M_p \le AH^p \min_{0\le q\le p} M_q M_{p-q},\ p = 0, 1, \dots .$$

$(M.3)$ (strong non-quasi-analyticity) There is a constant A such that

$$\sum_{q=p+1}^{\infty} \frac{M_{q-1}}{M_q} \le Ap\frac{M_p}{M_{p+1}},\ p = 1, 2, \dots .$$

Some results remain valid when (M_2) and (M_3) are replaced by the following weaker conditions:

$(M.2)'$ (stability under differential operators) There are constants A and H such that

$$M_{p+1} \le AH^p M_p,\ p = 0, 1, \dots .$$

$(M.3)'$ (non-quasi-analyticity)

$$\sum_{p=1}^{\infty} \frac{M_{p-1}}{M_p} < \infty, \ p = 1, 2, \ldots.$$

Remark B.0.1. Let the sequence $\{M_p\}_{p\in\mathbb{N}_0}$ be such that satisfies $(M.1)$ and $(M.2)'$. Let $S^{M_p,A}_{M_p,A}(\mathbb{R}^d)$ be defined by

$$S^{M_p,A}_{M_p,A}(\mathbb{R}^d) = \{f \in \mathcal{C}^\infty(\mathbb{R}^d) : \|x^m D^n f(x)\|_{L^\infty} \leq CA^{|m|+|n|} M_m M_n, \ \forall m, n \in \mathbb{N}_0^d\},$$

for some positive constant C, where $A = (A_1, ..., A_d) > 0$.

We define the Gelfand-Shilov space $S^{M_p}_{M_p}(\mathbb{R}^d)$ as an inductive limit of the spaces $S^{M_p,A}_{M_p,A}(\mathbb{R}^d)$ with respect to A:

$$S^{M_p}_{M_p}(\mathbb{R}^d) = \varinjlim_{A\to\infty} S^{M_p,A}_{M_p,A}(\mathbb{R}^d).$$

The corresponding dual space of $S^{M_p}_{M_p}(\mathbb{R}^d)$ is the space of ultradistributions of Roumier type:

$$(S^{M_p}_{M_p}(\mathbb{R}^d))' = \varprojlim_{A\to 0} S^{M_p,A}_{M_p,A}(\mathbb{R}^d).$$

Next, notice that the condition $(M.1)$ is equivalent to the assumption that the sequence

$$m_p = \frac{M_p}{M_{p-1}}, \ p \in \mathbb{N},$$

increases monotonically. Furthermore, if the sequence $m_p = M_p/M_{p-1}$, $p \in \mathbb{N}$, tends to infinite, then we define the associated function of M_p as (see [21, (0.14, p.29)]):

$$M(t) = \sup_{p\in\mathbb{N}_0} \ln \frac{t^p M_0}{M_p}, \ t \in (0, \infty).$$

It is a monotonically increasing continuous function which vanishes for sufficiently small $t > 0$ and increases more rapidly than $\ln t^p$ for any p as $t \to \infty$.

In this thesis, we considered only the Gevrey sequence

$$M_p = \{p!^\alpha\}_{p\in\mathbb{N}_0}, \text{ with } \alpha \geq 1 \text{ and } p \in \mathbb{N}_0.$$

Note that the Gevrey sequence satisfies the above conditions. By $M(\cdot)$ we denote the associated function of $\{p!^\alpha\}_{p\in\mathbb{N}_0}$.

We call an entire function $P : \mathbb{C}^d \to \mathbb{C}$,

$$P(z) = \sum_{n\in\mathbb{N}_0^d} c_n z^n,$$

an ultrapolynomial of class $\{p!^\alpha\}$ if for every $h > 0$ there exists $C > 0$ such that

$$|c_n| \leq C \frac{h^{|n|}}{|n|!^\alpha}.$$

Next, we state a sufficient and necessary conditions for existence of an ultrapolynomial of class $\{p!^\alpha\}$.

Proposition B.0.1. *([21, Proposition 4.5, p.58]) Suppose that M_p satisfies $(M.1)$. Then the following are equivalent conditions for entire functions:*

$$P(z) = \sum_{n \in \mathbb{N}_0^d} c_n z^n :$$

(i) For every constant $h > 0$ there exists $C > 0$ such that

$$|P(z)| \leq Ce^{M(h|z|)}, \; \forall z \in \mathbb{C}^d.$$

(ii) For every constant $h > 0$ there exists $C > 0$ such that

$$|c_n| \leq C \frac{h^{|n|}}{|n!|^{\alpha}}.$$

Definition B.0.1. ([21, Definition 3.11, p.53]) A continuous increasing function $\varepsilon(t)$ on $[0, \infty)$ which satisfies

$$\varepsilon(0) = 0 \quad \text{and} \quad \frac{\varepsilon(t)}{t} \to 0, \; \text{as } t \to \infty$$

is called a subordinate function.

Lemma B.0.1. *([21, Lemma 3.12], p.54) Suppose that M_p satisfies $(M.1)$ and that $\varepsilon(t)$ is a subordinate function. Then there is a sequence N_p of positive numbers which satisfies $(M.1)$ and the following properties:*

$$N(t) \geq M(\varepsilon(t)), \; 0 < t < \infty$$

and

$$\frac{m_p}{n_p} = \frac{M_p N_{p-1}}{M_{p-1} N_p} \to 0, \; as \; p \to \infty.$$

In particular, we have $M_p \prec N_p$ so that there is a subordinate function $\varepsilon'(t)$ such that

$$N(t) = M(\varepsilon'(t)).$$

Appendix C

Sobolev embedding theorem

Definition C.0.1. ([2, Definition 4.9, p. 83]) Ω satisfies the strong Lipschitz condition if there exist positive numbers δ and M, a locally finite open cover $\{U_j\}$ of bdry Ω (bdry stands for boundary) and for each j a real-valued function f_j of $n-1$ variables, such that the following condition hold

(i) For some finite R, every collection of $R+1$ of the sets U_j has empty intersection.

(ii) For every pair of points $x, y \in \Omega_\delta$ such that $|x-y| < \delta$, there exists j such that
$$(x, y) \in V_j = \{x \in U_j : \operatorname{dist}(x, \text{bdry } U_j) > \delta\}.$$

(iii) Each function f_j satisfies a Lipschitz condition with constant M: that is, if $\xi = (\xi_1, \ldots, \xi_{n-1})$ and $\rho = (\rho_1, \ldots, \rho_{n-1})$ are in $\mathbb{R}^{n-1}$, then
$$|f(\xi) - f(\rho)| \leq M|\xi - \rho|.$$

(iv) For some Cartesian coordinate system $(\zeta_{j,1}, \ldots, \zeta_{j,n})$ in U_j, $\Omega \cap U_j$ is represented by the inequality
$$\zeta_{j,n} < f_j(\zeta_{j,1}, \ldots, \zeta_{j,n-1}).$$

If Ω is bounded, the rather complicated set of conditions above reduce to the simple condition that Ω should have a locally Lipschitz boundary, that is, that each point x on the boundary of Ω should have a neighbourhood U_x whose intersection with bdry Ω should be the graph of a Lipschitz continuous function.

The Sobolev embedding theorem asserts the existence of embedding of Sobolev spaces
$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } 0 \leq |\alpha| \leq m\},\ m \in \mathbb{N},\ 1 \leq p \leq \infty$$
into Banach spaces of following types:

(i) $C^j_B(\Omega)$, the space of function having bounded, continuous derivatives up to order j on Ω normed by
$$\|u\|_{C^j_B(\Omega)} = \max_{0 \leq |\alpha| \leq j} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

(ii) $C^j(\overline{\Omega})$, the closed subspace of $C_B^j(\Omega)$ consisting of function having bounded, uniformly continuous derivatives up to order j on Ω with the same norm as $C_B^j(\Omega)$.

Theorem C.0.1. *([2, Theorem 4.12, p. 85]) Let Ω be a domain in $\mathbb{R}^n$ and for $1 \leq k \leq n$, let Ω_k be the intersection of Ω with a plain of dimension k in $\mathbb{R}^n$ (if $k = n$, then $\Omega_k = \Omega$). Let $j \geq 0$ and $m \geq 1$ be integers and let $1 \leq p \leq \infty$. Suppose Ω satisfies the strong local Lipschitz condition. Then the target space $C_B^j(\Omega)$ of the embedding*

$$W^{j+m,p}(\Omega) \hookrightarrow C_B^j(\Omega)$$

can be replaced with the smaller space $C^j(\overline{\Omega})$ and the embedding can be further refined as follows:

If $mp > n > (m-1)p$, then

$$W^{j+m,p}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega}) \text{ for } \quad 0 < \lambda \leq m - (n/p),$$

and if $n = (m-1)p$, then

$$W^{j+m,p}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega}) \quad \text{for } 0 < \lambda \leq 1. \tag{C.1}$$

Also, if $p = 1$ then (C.1) holds for $\lambda = 1$ as well.

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Short Biography

Smiljana Radojko Jakšić

Address:	Bulevar oslobodjenja 54, Novi Sad, Srbija
Telephone number:	+381 21 6336 348
e-mail:	smiljana.jaksic@gmail.com
Date and place of birth:	1982, Sremska Mitrovica, Srbija
Citizenship:	Srbija

Education: 2010

MSc degree in theoretical mathematics,
with grade-point average 10.0,
Faculty of Sciences,
University of Novi Sad,
MSc thesis: "*The Weyl transform in the spaces of distributions and ultradistributions*"
(serbian)

2005

Bachelor degree in mathematics-financial mathematics,
with grade-point average 9.73,
Faculty of Sciences,
University of Novi Sad

Work experience: 2009-today

teaching assistant,
Faculty of Forestry,
University of Belgrade

2008-2009

junior teaching assistant,
Faculty of Agriculture,
University of Novi Sad

Bibliography-papers:

S. Jakšić, B. Prangoski, *Extension theorem of Whitney type for* $\mathcal{S}(\mathbb{R}^d_+)$ *by the use of the Kernel Theorem*, Publ. Inst. Math. Beograd, 99(113)(2016), 59-65.

S. Jakšić, S. Pilipović, B. Prangoski, *G-type spaces of ultradistributions over* $\mathbb{R}^d_+$ *and the Weyl pseudo-differential operators with radial symbols*, arXiv:1602.04017.

Novi Sad, 2016 Smiljana Jakšić

Кратка Биографија

Смиљана Радојко Јакшић

Адреса: Булевар ослобођења 54, Нови Сад, Србија

Телефонски број: +381 21 6336 348

e-mail: smiljana.jaksic@gmail.com

Датум и место рођења: 1982, Сремска Митровица, Србија

Држављанство: Србија

Образовање:

2010

Магистар математичких наука,
са просеком 10.0,
Природно-математички факултет,
Универзитет у Новом Саду,
Магистарски рад:
"*Вејлова трансформација у просторима дистрибуција и ултрадистрибуција*"
(српски)

2005

Дипломирани математичар-математике финансија,
са просеком 9.73,
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Радно искуство: 2009-данас

асистент,
Шумарски факултет,
Универзитет у Београду

2008-2009

сарадник у настави,
Пољопривредни факултет,
Универзитет у Новом Саду

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Abstract: We study the expansions of the elements in $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$ with respect to the Laguerre orthonormal basis. As a consequence, we obtain the Schwartz kernel theorem for $\mathcal{S}(\mathbb{R}^d_+)$ and $\mathcal{S}'(\mathbb{R}^d_+)$. Also we give the extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}^d_+)$. Next, we consider the G-type spaces i.e. the spaces $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ and their dual spaces which can be described as analogous to the Gelfand-Shilov spaces and their dual spaces. Actually, we show the existence of the topological isomorphism between the G-type spaces and the subspaces of the Gelfand-Shilov spaces $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, $\alpha \geq 1$ consisting of "even" functions. Next, we show that the Fourier Laguerre coefficients of the elements in the G-type spaces and their dual spaces characterize these spaces through the exponential and sub-exponential growth of the coefficients. We provide the full topological description and the kernel theorem is proved. Also two structural theorems for the dual spaces of G-type spaces are obtained. Furthemore, we define the new class of the Weyl pseudo-differential operators with radial symbols belonging to the G-type spaces and their dual spaces. The continuity properties of this class of pseudo-differential operators over the Gelfand-Shilov type spaces and their duals are proved. In this way the class of the Weyl pseudo-differential operators is extended to the one with the radial symbols with the exponential and sub-exponential growth rate.
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Апстракт: Проучавамо развоје елемената из $\mathcal{S}(\mathbb{R}^d_+)$ и $\mathcal{S}'(\mathbb{R}^d_+)$ преко Лагерове ортонормиране базе. Као последицу добијамо Шварцову теорему о језгру за $\mathcal{S}(\mathbb{R}^d_+)$ и $\mathcal{S}'(\mathbb{R}^d_+)$. Такође, показујемо и Теорему Витнијевог типа за $\mathcal{S}(\mathbb{R}^d_+)$. Затим, посматрамо просторе G-типа и.е. просторе $G^\alpha_\alpha(\mathbb{R}^d_+)$, $\alpha \geq 1$ и њихове дуале који су аналогни са Гељфанд-Шиловим просторима и њиховим дуалима. Заправо, показујемо да постоји тополошки изоморфизам између простора G-типа и потпростора Гељфанд-Шилових простора $\mathcal{S}^{\alpha/2}_{\alpha/2}(\mathbb{R}^d)$, $\alpha \geq 1$ који садрже "парне" функције. Даље, доказујемо да Фурије Лагерови коефицијенти елемената из простора G-типа и њихових дуала карактеришу ове просторе кроз експоненцијални и суб-експоненцијални раста тих коефицијената. Описујемо тополошку структуру ових простора и дајемо Шварцову теорему о језгру. Такође, две структуралне теореме за дуале простора G-типа су добијене. Даље, дефинишемо нову класу Вејлових псеудо-диференцијалних оператора са радијалним симболима који се налазе у просторима G-типа и њиховим дуалима. Показана је непрекидност ове класе Вејлових псеудо-диференцијалних оператора на просторима Гељфанд-Шилова и на њиховим дуалима. На овај начин класа Вејлових псеудо-диференцијалних оператора је проширена на радијалне симболе који имају експоненцијални и суб-експоненцијални раст. рате.
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