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MICROPOLAR FLUID BETWEEN TWO COAXIAL CYLINDERS (NUMERICAL APPROACH)

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Abstract. The paper describes the flow of a suspension which is a mixture of two phases: liquid and solid granules. The continuum model with microstructure is introduced, which involves two independent kinematic quantities: the velocity vector and the micro-rotation vector. The physical analogy is based on the movement of the suspension between two coaxial cylinders. The inner cylinder is stationary and the outer one rotates with constant angular velocity. This physical analogy enabled a mathematical model in a form of two coupled differential equations with variable coefficients. The aim of the paper is to present the numerical aspect of the solution for this complex mathematical model. It is assumed that the solid granules are identically oriented and that under the influence of the fluid they move translationally or rotate around the symmetry axis but the direction of their symmetry axes does not change. The solution was obtained by the ordinary finite difference method, and then the corresponding sets of points (nodes) were routed by interpolation graphics.

1. Introduction

Mechanics of suspensions is a part of a general physical-chemical sphere of knowledge about dispersions. In this paper, it is supposed that the dispersive phase is a viscous incompressible fluid and that the disperse phase consists of a large number of small solid particles (granules) suspended in the fluid. The sizes of particles are assumed to be of the same order as the distances between the nearest particles.

Eringen and Suhubi [**6**] introduced a micro polar continuum and micro polar fluid models characterized by couple stress and a nonsymmetrical stress tensor. This theory comprises two independent kinematic quantities: the velocity vector and the micro-rotation vector. Another approach is based on negative electro-rheological responses induced by micro-particle electro-rotation in two-dimensional model (Huang et al [**9**]). All non-classical continuum theories for solid and fluent

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continua were described by Surana et al. [**18**]. The analyzed theories ranged from the first non-classical continuum theory with Jacobian of deformation as well as Cosserat rotations in the conservation to Eringen's theory and its modifications [**3**, **14**–**17**]. The micropolar fluid model can, among other applications, be used to describe the motion of a suspension as a mixture of two phases [**7**, **13**]. The basic phase of the suspension is a fluid, whereas the dispersive phase consists of solid particles. The theory of micropolar fluids initiated by Eringen shows some microscopic effects arising from the local structure and micro motion of the fluid elements. Further, they can sustain couple stresses and include classical Newtonian fluid as a special case.

There are many applications of this theory such as in medicine (peristalsis–an inherent property of many tubular organs of the human body, reported by Abeer et al. [**1**]). Furthermore, the study of such fluids has applications in a number of processes that occur in industry such as extraction of crude oil from petroleum products, aerodynamic heating, electrostatic precipitation, solidification of liquid crystals, cooling of metallic plate in a bath, exotic lubricants, colloidal and suspension solutions [**11**]. Also, the effects of local rotary inertia and couple stress represent behavior of many industrially important fluids such as: paints, biological fluids, polymers, colloidal fluids, suspension fluids, etc. [**12**].

The model of micropolar fluid represents fluids consisting of rigid randomly oriented (or spherical) particles suspended in a viscous medium where the deformation of the particles is ignored. The fluids containing certain additives, some polymeric fluids and animal blood are examples of micropolar fluids. The mathematical theory of equations of micropolar fluids and applications of these fluids in the theory of lubrication and in the theory of porous media was introduced by Lukaszewicz [**10**].

The aim of our paper is to present the numerical aspect of the solution for the complex mathematical model based on the physical model movement of suspension which may have one type of the content mentioned in the literature above. With a numerical solution the values of velocities can be determined for each value of the radial coordinate.

Nomenclature

$v(r)$	velocity of suspension (m/s);
$w(r)$	velocity of microrotation of suspension (m/s);
r	radial coordinate (m);
α_1	constant;
α_2	constant;
B	constant;
ω	angular speed $(1/s)$;
Δ	numerical step of integration (m);
n	number of steps of numerical integration;
A	square matrix;
$\mathbf{x}$	matrix column;
$\mathbf{y}$	matrix column.

2. Mathematical model

Figure 1 shows two coaxial cylinders: the inner one, which is stationary, and the outer rotating one with constant angular velocity (ω), while the suspension of certain physical properties moves between them. The radii of these cylinders are the internal cylinder (r_0) and the external cylinder (r_k).

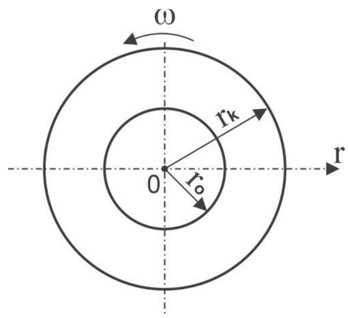

FIGURE 1. A simplified functional diagram of the movement of the suspension between two coaxial cylinders

The mathematical model of the flow of the suspension describes the velocity field of the flow of suspension, describing the velocity field of the movement of the suspension (v) and the velocity field of the microrotation of the suspension (w), depending on the radial coordinate (r) and defined by a coupled system of two ordinary linear differential equations of the second order with variable coefficients, which have the following form [**4**, **7**, **19**]:

$$r^2 \frac{d^2 v}{dr^2} + r \frac{dv}{dr} - v - \frac{\alpha_1}{1+\alpha_1} r^2 \frac{dw}{dr} = 0, \tag{2.1}$$

$$\alpha_2 r \frac{dv}{dr} + \alpha_2 v + r \frac{d^2 w}{dr^2} + \frac{dw}{dr} - 2\alpha_2 r w = 0, \tag{2.2}$$

where: $\alpha_1 = \text{const} > 0$ and $\alpha_2 = \text{const} > 0$. These constants are general and depend on the type of suspension. They included thermo-physical parameters of the liquid phase and solid granules, such as: the density of both components, fluid viscosity, heat mass capacity and the material constants.

The mathematical model presented with equations (2.1) and (2.2) originates from the equations of continuity and first stress momentum, while the energy equation with temperature influence was not taken into the consideration. It was assumed that the angular velocity of the external cylinder is high enough to cause the adiabatic process in which there is no heat transfer between the suspension and walls of cylinders. Boundary conditions for equations (2.1) and (2.2) are:

$$v(r)|_{r=r_0} = v_0 = 0, \tag{2.3}$$

$$v(r)|_{r=r_k} = v_k = r_k \omega, \tag{2.4}$$

$$w(r)|_{r=r_0} = w_0 = 0, \tag{2.5}$$

$$w(r)|_{r=r_k} = w_k = 0. \tag{2.6}$$

Before the very solving of the mathematical model, a reduction of the order of equations (2.1) and (2.2) will be introduced from the second to the first order, which is known as the "Koschi method". In that way the following dependent variables will be defined:

$$a(r) = v(r), \tag{2.7}$$

$$b(r) = \frac{dv(r)}{dr}, \tag{2.8}$$

$$c(r) = w(r), \tag{2.9}$$

$$d(r) = \frac{dw(r)}{dr}. \tag{2.10}$$

Equations (2.1) and (2.2), having in mind relations (2.7)–(2.10), are now a common system of four linear first-order differential equations with variable coefficients, which are gathered in one place as follows:

$$\frac{da(r)}{dr} = b(r), \tag{2.11}$$

$$\frac{db(r)}{dr} = \frac{1}{r^2}\, a(r) - \frac{1}{r}\, b(r) + \frac{\alpha_1}{1+\alpha_1}\, d(r), \tag{2.12}$$

$$\frac{dc(r)}{dr} = d(r), \tag{2.13}$$

$$\frac{dd(r)}{dr} = -\frac{\alpha_2}{r}\, a(r) - \alpha_2 b(r) + 2\alpha_2 c(r) - \frac{1}{r}\, d(r). \tag{2.14}$$

Boundary conditions (2.3)–(2.6), according to the changes in the structure of equations (2.1) and (2.2), become:

$$a(r)|_{r=r_0} = a(r_0) = a_0 = 0, \tag{2.15}$$

$$a(r)|_{r=r_k} = a(r_k) = r_k \omega, \tag{2.16}$$

$$c(r)|_{r=r_0} = c(r_0) = c_0 = 0, \tag{2.17}$$

$$c(r)|_{r=r_k} = c(r_k) = c_k = 0. \tag{2.18}$$

A complete numerical algorithm of solving the system of equations (2.11)–(2.14) will begin by defining a "network", i.e. with a division of the boundary contour in the sequence, according to the adopted step of integration (Δ). Boundary is defined as the "integration network" and is shown in Figure 2.

The integration step size will be:

$$\Delta = \frac{r_k - r_0}{n}, \tag{2.19}$$

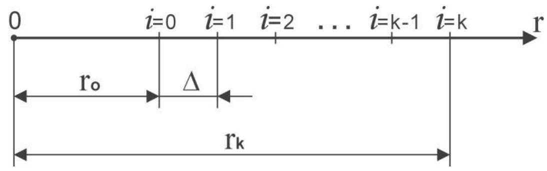

FIGURE 2. The integration network defined to solve the system of equations (2.11)–(2.14)

where the value n is the number of steps along the boundary contour.

Derivations of values $a(r)$, $b(r)$, $c(r)$ and $d(r)$, changed by ordinary finite differences, were defined as follows:

$$\frac{da}{dr} \approx \frac{a_i - a_{i-1}}{\Delta}, \tag{2.20}$$

$$\frac{db}{dr} \approx \frac{b_i - b_{i-1}}{\Delta}, \tag{2.21}$$

$$\frac{dc}{dr} \approx \frac{c_i - c_{i-1}}{\Delta}, \tag{2.22}$$

$$\frac{dd}{dr} \approx \frac{d_i - d_{i-1}}{\Delta}. \tag{2.23}$$

Substituting expressions (2.20)–(2.23), into equations (2.11)–(2.14), we obtain:

$$a_i - a_{i-1} - \Delta b_{i-1} = 0, \tag{2.24}$$

$$b_i - b_{i-1} - \frac{\Delta}{r_{i-1}^2} a_{i-1} + \frac{\Delta}{r_{i-1}} b_{i-1} - \frac{\alpha_1}{1+\alpha_1} \Delta d_{i-1} = 0, \tag{2.25}$$

$$c_i - c_{i-1} - \Delta d_{i-1} = 0, \tag{2.26}$$

$$d_i - d_{i-1} + \frac{\alpha_2}{r_{i-1}} \Delta a_{i-1} + \alpha_2 \Delta b_{i-1} - 2\alpha_2 \Delta c_{i-1} + \frac{\Delta}{r_{i-1}} d_{i-1} = 0. \tag{2.27}$$

The upper numerical algorithm for the calculation of coefficients a, b, c, d was applied to all points $r_i = r_0 + i\Delta$ on the "net" in the following range:

$$i = 1, 2, \ldots, k. \tag{2.28}$$

Since the values a_0, c_0, a_k and c_k are known from boundary conditions (2.15)–(2.18), equations (2.24)–(2.27) give us a system of $4k$ linear algebraic equations with $4k$ unknowns:

$$\begin{gathered} b_0, \quad d_0, \\ a_i, \quad b_i, \quad c_i, \quad d_i \quad \text{for} \quad i = 1, 2, \ldots, k-1, \\ b_k, \quad d_k. \end{gathered}$$

This system of linear algebraic equations can be rewritten in matrix form:

$$Ax = \mathbf{y}, \tag{2.29}$$

where

$$A = \begin{pmatrix} (A')^0 & I & O & \dots & O & O & O' \\ O' & A^1 & I & \dots & O & O & O' \\ O' & O & A^2 & \dots & O & O & O' \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O' & O & O & \dots & A^{k-2} & I & O' \\ O' & O & O & \dots & O & A^{k-1} & I' \end{pmatrix}$$

$$O = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad O' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad I' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

$$(A')^0 = \begin{pmatrix} -\Delta & 0 \\ \left(\frac{\Delta}{r_0} - 1\right) & -\frac{\alpha_1 \Delta}{1+\alpha_1} \\ 0 & -\Delta \\ \alpha_2 \Delta & \left(\frac{\Delta}{r_0} - 1\right) \end{pmatrix},$$

$$A^i = \begin{pmatrix} -1 & -\Delta & 0 & 0 \\ -\frac{\Delta}{r_{i-1}^2} & \left(\frac{\Delta}{r_{i-1}} - 1\right) & 0 & -\frac{\alpha_1 \Delta}{1+\alpha_1} \\ 0 & 0 & -1 & -\Delta \\ \frac{\alpha_2 \Delta}{r_{i-1}} & \alpha_2 \Delta & -2\alpha_2 \Delta & \left(\frac{\Delta}{r_{i-1}} - 1\right) \end{pmatrix}, \quad i = 1, 2, \dots, k-1,$$

$$\mathbf{x} = \begin{pmatrix} b_0 \\ d_0 \\ a_1 \\ b_1 \\ c_1 \\ d_1 \\ \vdots \\ a_{k-1} \\ b_{k-1} \\ c_{k-1} \\ d_{k-1} \\ b_k \\ d_k \end{pmatrix}, \qquad \mathbf{y} = \begin{pmatrix} e \\ f \\ g \\ h \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \tilde{k} \\ 0 \\ l \\ 0 \end{pmatrix}, \qquad \begin{aligned} e &= a_0, \\ f &= \frac{\Delta a_0}{r_0^2}, \\ g &= c_0, \\ h &= -\frac{\alpha_2 \Delta a_0}{r_0} + 2\alpha_2 \Delta c_0, \\ \tilde{k} &= -a_k, \\ l &= -c_k. \end{aligned}$$

Note that the square matrix A and the matrix column $\mathbf{y}$ contain known parameters while the matrix column $\mathbf{x}$ contains unknown quantities. The unknown column matrix $\mathbf{x}$ can be obtained by solving the matrix linear algebraic equation (2.29) as follows:

$$\mathbf{x} = A^{-1}\mathbf{y}. \tag{2.30}$$

The main problem in solving the matrix equation (2.29) is to find the inverse matrix A^{-1} of the square matrix A and then multiply it with the matrix column $\mathbf{y}$. A higher number of numerical integration steps produces greater dimensions of the matrixes A and $\mathbf{y}$ and, therefore, the problem becomes more complex. The accuracy of the numerical solutions grows as the number of steps increases, so from that point of view it is preferable for the matrixes A and $\mathbf{y}$ to be larger in size. The requirements in terms of accuracy of the solutions are contradictory and, in fact, limited by the availability of appropriate hardware and software of digital computers to calculate the values in the matrix column $\mathbf{x}$.

3. Results and discussion

In this section. the numerical simulation was performed without experimental verification, as it was complicated to measure the velocity of the microrotation of the suspension w and to determine the thermo-physical parameters included in the constants α_1 and α_2. The presented numerical procedure was performed in the software package EXCEL.

The calculated values in the matrix column $\mathbf{x}$ are further used for the construction of graphs presented in Figures 3 to 6.

Finally, it should be necessary to form the interpolation dependences for the values: $v(r)$, $dv(r)/dr$, $w(r)$ and $dw(r)/dr$ as a function of radial coordinate r, and draw their graphics. HeThe results of the previous analysis will be presented here in case $\alpha_1 = 10$, $\alpha_2 = 10$, $\omega = 100$, $r_k = r_{10} = 0.0048$, $k = 10$, $n = k = 10$,

$$\Delta = \frac{r_k - r_0}{n} = \frac{r_{10} - r_0}{n} = 0.00008.$$

It should be noted that since the number of steps of the numerical integration was selected to be ten ($n = 10$), the dimensions of a square matrix A were 40×40 and dimensions of matrix columns $\mathbf{x}$ and $\mathbf{y}$ were 40×1.

Graphs for values: $v(r)$, $dv(r)/dr$, $w(r)$ and $dw(r)/dr$ are shown in Figures 3 to 6.

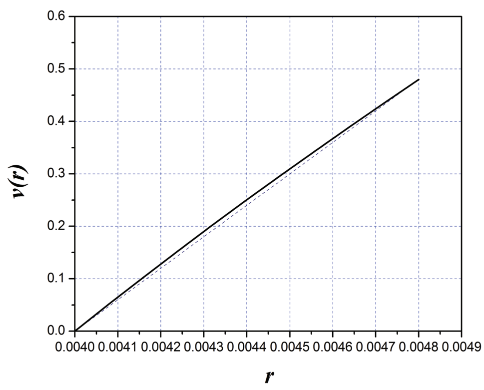

FIGURE 3. Velocity of the suspension $v(r)$ depending on the radial coordinate r

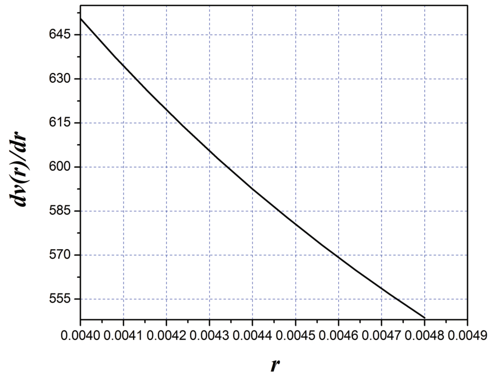

FIGURE 4. Graph size $\frac{dv(r)}{dr}$ depending on the radial coordinate r

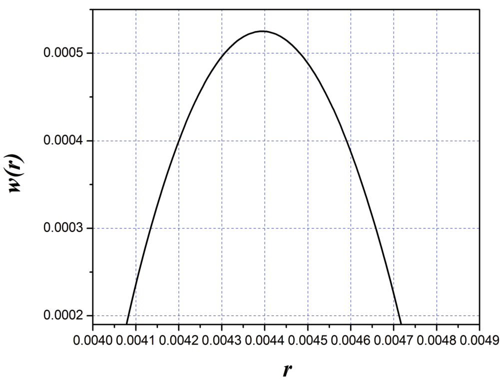

FIGURE 5. Velocity of microrotation of the suspension $w(r)$ depending on the radial coordinate r

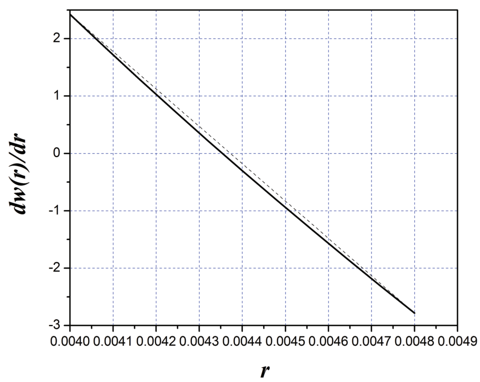

FIGURE 6. Graph size $\frac{dw(r)}{dr}$ depending on the radial coordinate r

The confirmation of the correctness of the whole procedure is the compliance of values: $v(r)$, $dv(r)/dr$, $w(r)$ and $dw(r)/dr$. Since the value r at the interval $[r_0, r_k]$ increases monotonously, its first derivative must be greater than zero, which is evident on the graph $dv(r)/dr$ (Figure 4). In addition, at the interval $[r_0, r_k]$, where the velocity of microrotaion $w(r)$ increases (Figure 5), its first derivative $dw(r)/dr$ is greater than zero (Figure 6), where the size of $w(r)$ decreases, its first derivative $dw(r)/dr$ is less than zero, which is the expected behavior. Consequently, the maximum value of $w(r)$ corresponds to the zero value of $dw(r)/dr$.

Finally, it should be mentioned that the problem was solved in the analytical form [**8**, **19**]. The disadvantage of these analytical solutions is their complexity. In order to obtain the values of the velocity of suspension v and the velocity of microrotation w, the coefficients that consist of Bessel's functions must be calculated first. That is why the numerical solutions have a more practical application, although it showed good agreement with the results of the analytical procedure [**8**,**19**]. Furthermore, Agarwal and Dhanpal [**2**] obtained a numerical solution for the micropolar fluid flow with heat transfer between two coaxial porous circular cylinders. The different curves for the velocity of microrotation were given depending on the porosity and axial pressure gradient. Good agreement was achieved with Figure 5, in the case when there is no porosity and axial pressure gradient. In paper [**2**], besides the equations of continuity and momentum, the energy equation was taken into consideration and through it the temperature distribution was also obtained.

4. Conclusions

The results of the numerical procedure of solving the system of differential equations (2.1) and (2.2) are the graphic values: v, dv/dr, w and dw/dr that show excellent accuracy and agreement with the expected behavior of suspension. Furthermore, the numerical solution for the model can be used in predicting the velocity field of the movement of the suspension v and the velocity field of the micro rotation of the suspension w. This is of great interest for technical and industry practice.

In the end, it should be mentioned that the results presented here are in complete accordance with the results presented in papers [**8**, **19**]. Differences in the values of $v(r)$ and $w(r)$ between the analytical and numerical solutions are negligible, i.e. differences in the case of velocity $v(r)$ are on the order of 10^{-3}, and in the case of velocity $w(r)$ they are on the order of 10^{-6}.

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МИКРОПОЛАРНИ ФЛУИД ИЗМЕЂУ ДВА КОАКСИЈАЛНА ЦИЛИНДРА (НУМЕРИЧКИ ПРИСТУП)

Резиме. У раду се разматра проток суспензије која је мешавина две фазе: течности и чврстих гранула. Уведен је модел континуума са микроструктуром који укључује две независне кинематичке величине: вектор брзине и вектор микро ротације. Физичка аналогија се заснива на кретању суспензије између два коаксијална цилиндра. Унутрашњи цилиндар је непокретан, а спољашњи ротира константном угаоном брзином. Ова физичка аналогија је омогућила математички модел у виду две спрегнуте диференцијалне једначине са промељивим коефицијентима. Циљ рада је био да представи нумерички аспект решења овог сложеног математичког модела. Претпостављено је да су чврсте грануле идентично оријентисане и да се под утицајем флуида (течности) померају транслаторно или ротирају око осе симетрије, али се правац њихових оса симетрије не мења. Нумеричко решење је добијено методом обичних коначних разлика, а затим су кроз одговарајуће скупове тачака (чворове) провучени интерполациони графици.

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